Statistical mechanics of CSMA networks

Sunil Srinivasa and Martin Haenggi
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, USA
Email: {*ssriniv1, mhaenggi*}*@nd.edu*

***Abstract*—This paper is intended to depict the rich behavior even for simple protocols such as CSMA.**

I. Introduction

An ad hoc network is formed by deploying nodes that possess self-organizing capabilities and typically consists of several source-destination pairs communicating wirelessly with each other in a decentralized fashion. On account of such intricate interactions, ad hoc networks evade familiar link-based decompositions; studying them using traditional methods such as information theory becomes intractable and hence has yielded little in the way of results [1]. This has motivated researchers to turn to other branches of study, to obtain ideas and methodologies that help better understand and characterize the dynamical behavior of multihop networks. Of late, statistical physics has, in particular, captured the attention of the research community since it contains a rich collection of mathematical tools and methodologies for studying interacting many-particle systems [2]-[4].

II. System Model

In this paper, we treat the wireless ad hoc network as an infinite geometric graph $G = (V, E)$, with a set of *vertices* V (representing the node locations) and a set of *edges* E (representing possible links between nodes in the network). The vertices of the graph $v \in V$ are assumed to be distributed as a *motion-invariant*[1] point process $\Phi = \{x_i\}$ on the plane $\mathbb{R}^2$ with intensity λ. Without loss of generality, we take $\lambda = 1$. Furthermore, each node has a certain communication range R; an edge (or a link) may be formed only between two nodes that are in each other's communication range.

In this paper, we focus on the carrier sense multiple access (CSMA) channel access scheme, in which a node verifies the absence of other traffic before transmitting its own packets. We also assume that all nodes use the same wireless channel. Thus, interference in the network permits only certain configurations of edges: if nodes i and j communicate with each other, all the other nodes within the communication ranges of either node must remain silent (for example, see Fig. 1). The edges are said to be *undirected* in that both nodes involved in any link can exchange information with each other, i.e., whenever node i can receive a packet from node j, node j can also do so from node i.

[1]A point process is said to be motion-invariant if its both stationary (invariant to translations) and isotropic (invariant to rotations).

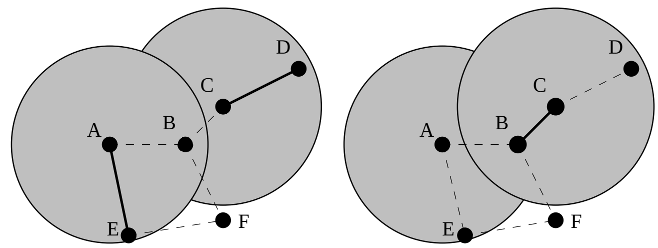

Fig. 1. Two possible link configurations for a portion of the network (with 6 nodes). Nodes in the communication range of already-communicating nodes must remain silent. Active edges are depicted via solid lines while inactive edges are depicted via dashed lines. The communication ranges of nodes A and C are also shown.

Time is taken to be slotted. At $t = 0$, suppose that no edges in the (network) graph are present. In every time slot, a new set of edges may be added to the network (if they do not interfere with other transmissions), or a pair of already-communicating nodes may choose to end their connection. We assume that both rates happen with given rates: each edge that is inactive is activated (if possible) at unit rate, and each edge that is active becomes inactive at rate $1/\rho$. Thus, the probability of finding any given allowed configuration of links with N *active edges* is $Z^{-1}\rho^N$, with Z being the partition function:

$$Z = \sum_{c \in \mathcal{C}} \rho^{|c|},$$

where $c \in \mathcal{C}$ is a configuration of the network with $|c|$ active edges.

Clearly, the larger ρ is, larger the average number of edges will be; thus the network throughput is increased, which is desirable. However, larger the ρ, the network is stuck in a given configuration for a long period of time, which is undesirable from a fairness-standpoint. Also, the time taken by the network to reach steady state increases. Evidently, there exists a tradeoff.

In this paper, we study the following problems for a CSMA network:

1) Given a geometric graph with a certain degree distribution, we evaluate the probability that an arbitrarily chosen node is involved in an active link. We consider the special cases of a lattice graph and a uniformly random graph.
2) We evaluate the average throughput of the network at steady state, and study its dependence on the link

deactivation rate ρ. We then analytically derive the optimal ρ that maximizes the fairness (will be defined later) of the CSMA protocol.

All the results in this paper are for an "average" network, that is the one obtained by averaging over all possible node configurations and transmitter sets.

III. A Mean-Field Formulation on A Rooted Tree

In this section, we solve the above three problems for a *rooted tree*[2] as a good approximation to a random graph. In the subsequent sections, we validate the accuracy of these results via simulations.

We consider an arbitrary node R in the network and regard it as the root node of a tree T. Also, suppose the degree distribution[3] of this tree be given by $p(k)$. We now identify three possible states for the root node of T.

1) First, the root node can be involved in an active edge with one of its *daughters* - we call this the "active" state. Note that this state prevents the other daughter nodes from being active.
2) Second, one (or more) of the daughter nodes is involved in an active link - we call this the "locked" state. Under this state, the root itself is inactive, and cannot establish edges
3) Third, neither the root node nor any of its daughters is involved in an active link - we call this the "free" state.

Let the *partition function* of this tree for the active, locked and free states be Z_1, Z_2 and Z_3 respectively.

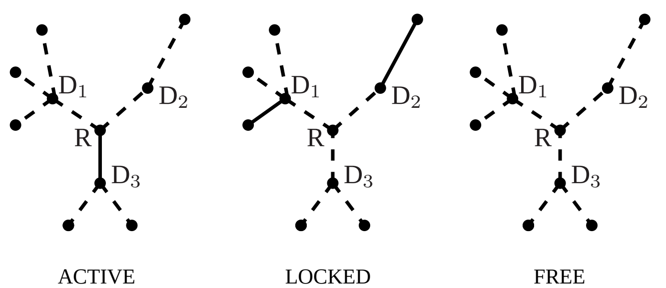

Fig. 2. An arbitrary node in the network R (designated as the root node) with degree $k = 3$; It has three daughters D_1, D_2 and D_3 with degrees 3, 1 and 2 respectively. We identify three possible states for it: active, locked and free.

Next, we construct a tree of a *higher order*. Accordingly, with probability $p(k)$, we take k such trees and join them together to make a tree of one higher level. Let Z_1', Z_2' and Z_3', respectively denote the partition functions for the larger (constructed) tree. Then the following equalities hold:

[2]A tree is a connected cycle-free graph in which any two nodes are connected by exactly one simple path. A rooted tree is a graph in which one vertex has been designated the root; the edges which connect the root node to its daughters have a natural orientation, towards or away from the root.

[3]The degree of a node is the number of edges emanating out of it. $p(k)$ simply represents the fraction of nodes with degree k

- For the root of the constructed tree to be active, one of the k daughters must be free to be involved in an edge with, and all other daughters must be locked or free (they cannot be active since the parent cannot otherwise be involved in an edge with the free daughter). So,

$$Z_1' = \sum_{k=1}^{\infty} \rho k Z_3 (Z_2 + Z_3)^{k-1} p(k).$$

- For the root of the constructed tree to be locked, at least one daughter must be active; we consider all possible states of the daughters, and subtract off those states with no daughters active. Accordingly,

$$Z_2' = \sum_{k=1}^{\infty} \left[(Z_1 + Z_2 + Z_3)^k - (Z_2 + Z_3)^k \right] p(k)$$

- For the constructed tree to be free, each of the n daughters of the new parent must be either locked or free (they cannot be active otherwise the new parent cannot be free). Thus,

$$Z_3' = \sum_{k=1}^{\infty} (Z_2 + Z_3)^k p(k).$$

In order to solve for the solution, we take that the partition functions evaluate to the same value for the original and the constructed trees, i.e., $Z_i = Z_i'$, for $i = 1, 2, 3$. The values of Z_1, Z_2 and Z_3 may then be obtained via numerical evaluation. The probabilities of finding any arbitrary node in the active, locked and free states are, respectively,

$$\frac{Z_1}{Z_1 + Z_2 + Z_3}\ ,\ \frac{Z_2}{Z_1 + Z_2 + Z_3} \text{ and } \frac{Z_3}{Z_1 + Z_2 + Z_3}.$$

Furthermore, the probability of a randomly chosen node being involved in an active edge is simply the probability of it being active, i.e., $Z_1/(Z_1 + Z_2 + Z_3)$.

In this paper, we consider two special cases: lattice networks and Poisson networks.

A. Lattice Networks

In this subsection, we consider the problem for a lattice network, where the coordination number for each node is the same; we take it to be equal to k, i.e., $p(k) = \delta(k)$, where $\delta(\cdot)$ denotes the delta function.

In other words, we obtain

$$\begin{aligned} Z_1 &= \rho k Z_3 (Z_2 + Z_3)^{k-1} \\ Z_2 &= (Z_1 + Z_2 + Z_3)^k - (Z_2 + Z_3)^k \\ Z_3 &= (Z_2 + Z_3)^k. \end{aligned} \tag{1}$$

We may simplify this further by normalizing the partition functions. Simply take $Z_i = Z_i/(Z_2 + Z_3)^k$, for $i = 1, 2, 3$: we get $Z_3 = 1$, while

$$\begin{aligned} Z_1 &= \frac{\rho k}{(Z_2 + 1)} \\ Z_2 &= \frac{(Z_1 + Z_2 + 1)^k - (Z_2 + 1)^k}{(Z_2 + 1)^k} \end{aligned} \tag{2}$$

Asymptotics:
When $\rho >> 0$, it is possible to obtain closed-form analytical solutions. Indeed, we have $Z_1 \gg Z_2 \gg Z_3 = 1$; so that $Z_1 + Z_2 + 1 \approx Z_1$ and $Z_2 + 1 \approx Z_2$. Then, (2) reduces to

$$\begin{aligned} Z_1 &\approx \frac{\rho k}{(Z_2)} \\ Z_2^{k+1} &\approx Z_1^k \end{aligned} \tag{3}$$

This leads to the following solution:

$$\begin{aligned} Z_1 &\approx (\rho k)^{(k+1)/(2k+1)} \\ Z_2 &\approx (\rho k)^{k/(2k+1)} \\ Z_3 &= 1. \end{aligned} \tag{4}$$

Furthermore, the probability that a given node being involved in an active link approximately equals $k/(2k+1)$.

Special Case: $k = 1$ We now consider the case of a line network.

B. Uniformly Random Networks

Lemma 3.1: Consider a PPP. The number of connected neighbors to any node follows a homogeneous PPP with density $\lambda\pi R^2$.

References

[1] J. Andrews, N. Jindal, M. Haenggi, R. Berry, D. Guo, M. Neely, S. Weber, S. Jafar and A. Yener, "Rethinking information theory for mobile ad hoc networks," *IEEE Communications Magazine*, Vol. 46, pp. 94-101, Dec. 2008.

[2] R. C. Alamino and D. Saad, "Statistical mechanics analysis of LDPC coding in MIMO Gaussian channels," *Journal of Physics A: Mathematical and Theoretical*, Vol. 40, No. 41, pp. 12259-12279, Oct 2007.

[3] K. Takeuchi and T. Tanaka, "Statistical-mechanics-based analysis of multiuser MIMO channels with linear dispersion codes," *Journal of Physics: Conference Series*, Vol. 95, Iss. 1, Jan. 2008.

[4] D. Guo and S. Verdu, "Randomly spread CDMA: asymptotics via statistical physics," *IEEE Transactions on Information Theory*, Vol. 51, pp. 1261-1282, Apr. 2005.