A Statistical Mechanics-Based Framework to Analyze Ad Hoc Networks with Random Access

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Abstract—Characterizing the performance of ad hoc networks is one of the most intricate open challenges; conventional ideas based on information-theoretic techniques and inequalities have not yet been able to successfully tackle this problem in its generality. Motivated thus, we promote the totally asymmetric simple exclusion process (TASEP), a particle flow model in statistical mechanics, as a useful analytical tool to study ad hoc networks with random access. Employing the TASEP framework, we first investigate the average end-to-end delay and throughput performance of a linear multihop flow of packets. Additionally, we analytically derive the distribution of delays incurred by packets at each node, as well as the joint distributions of the delays across adjacent hops along the flow. We then consider more complex wireless network models comprising intersecting flows, and propose the partial mean-field approximation (PMFA), a method that helps tightly approximate the throughput performance of the system. We finally demonstrate via a simple example that the PMFA procedure is quite general in that it may be used to accurately evaluate the performance of ad hoc networks with arbitrary topologies.

Index Terms—Ad hoc networks, throughput, end-to-end delay, statistical mechanics, network topology.

1 Introduction

1.1 Motivation

An ad hoc network is formed by deploying nodes that possess self-organizing capabilities and typically consists of several source-destination pairs communicating wirelessly with each other in a decentralized fashion. In order to conserve energy, yet efficiently deliver packets to distant nodes, routing is often performed in a multihop fashion, wherein relay nodes assist in the flow of packet traffic from the sources to the destinations. Characteristically, the multihop nature of packet transmissions causes interweaving of traffic flows, resulting in strong correlations, or interdependencies between the activities of the nodes.

For instance, since a traffic flow is relayed across several hops, the packet arrival processes at the nodes (and hence, the departure processes) are coupled with each other. Thus, the end-to-end delay in multihop networks, determined by the joint distribution of the successive delays of a packet traversing multiple nodes may hardly be expressed in a product form. Likewise, owing to the existence of relay nodes that serve multiple packet flows, the throughputs of the various flows in the network are correlated with each other.

On account of such intricate interactions, ad hoc networks evade familiar link-based decompositions; studying them using traditional methods such as information theory becomes intractable and hence has yielded little in the way of results [1]. This has motivated researchers to turn to other branches of study, to obtain ideas and methodologies that help better understand and characterize the dynamical behavior of multihop networks. Of late, statistical physics has, in particular, captured the attention of the research community since it contains a rich collection of mathematical tools and methodologies for studying interacting many-particle systems [2], [3], [4].

Along similar lines, we employ ideas from the *totally asymmetric simple exclusion process* (TASEP) [5], a subfield in statistical mechanics, to analyze multihop networks. The focus of this work is the investigation of the end-to-end metrics, delay and throughput of ad hoc networks with random access, taking into account the correlations in the system. Our main contributions are the following:

1. First, we consider a linear network model fed by a single source and characterize its average delay and throughput performances. Additionally, we analytically derive a) the probability mass functions (pmfs) of the delays incurred by packets at each node along the flow and b) the joint pmfs of the packet delays across adjacent nodes in the line network.
2. Second, we consider more complex ad hoc network models comprising intersecting packet flows. We introduce an elegant technique, the *partial mean-field approximation* (PMFA), which we employ to tightly approximate the throughput (and end-to-end delay) performance of such systems. We also demonstrate via a simple example on how to use the PMFA approach to accurately study networks with arbitrary topologies.

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1.2 Related Work

Most earlier attempts at analyzing ad hoc networks have neglected the correlations in the system for tractability. An approximation commonly used in this regard is *Kleinrock's Independence Assumption* [6]. Accordingly, for a densely connected network involving Poisson arrivals and uniform loading among source-destination pairs, the queues at each link in the network behave independently regardless of the interaction of traffic across different links. Kleinrock's approximation has been used to characterize the delay performance of wireless systems (see for example [7], [8]). Under general scenarios, however, this approximation may be very loose; the correlations in the system cannot be neglected.

Much of the prior work on the performance analysis of multihop networks has also focused on very small networks, (e.g., two-relay [9] or three-relay networks [10]). Their results, however, do not directly extend to larger networks. More recently, discrete-time queuing theory has been applied to the study of end-to-end delay [11] and throughput [12] performances of ad hoc networks. The authors, however, focus specifically on a linear multihop network model fed with a single flow, and do not consider intersecting flows. To the best of our knowledge, this is the first attempt at studying the throughput performances of ad hoc networks with arbitrary topologies.

The rest of the paper is organized as follows: Section 2 outlines the system and channel model. Section 3 provides an overview of the TASEP particle flow model, as well as the matrix product ansatz (MPA), the analytical tools that we use extensively for our analysis. In Section 4, we consider a wireless line network model, and characterize its delay and throughput performances. Section 5 introduces the PMFA framework, which we use to analyze the throughput performances of more complex network topologies. Section 6 concludes the paper.

2 System Model

We consider an ad hoc network comprising a set of source nodes intent to deliver packets to a set of destinations over an infinite duration of time in a multihop fashion. We study several different network topologies in this paper; the specifics of each topology will be described in its corresponding analysis section. Time is slotted to the duration of a packet, and packet transmissions occur at slot boundaries. No power control is employed, and the transmit power at each node is taken to be unity.

2.1 Channel Access Scheme

For analytical tractability, we consider a modified version of the traditional TDMA MAC scheme which we call *randomized-TDMA (r-TDMA)*. In r-TDMA, the transmitting node in each time slot is simply chosen uniformly randomly from the set of *all nodes in the network* instead of being picked in an deterministic fashion (as in conventional TDMA).

The r-TDMA scheme may also be viewed as a time-slotted version of the carrier sense multiple access (CSMA) protocol since in each time slot, only a single transmitter node gains the right to access the wireless channel. The only difference between slotted CSMA and r-TDMA is that in r-TDMA, nodes not having packets in their buffers may also be scheduled for transmission in some time slots. This, however, is equivalent to simply "stretching" the time axis. Also, note that the r-TDMA protocol does not entail spatial reuse. However, in small networks (which we primarily consider in this paper), spatial reuse is impractical, and the performance of the r-TDMA-based network is quite good (compared to other MAC schemes).

Owing to the presence of (at most) a single transmitter in each slot, there is no interference in the system; the probability of a successful transmission across a link, denoted by p_s, is dictated by the SNR model, i.e., $p_s = \mathbb{P}(\mathrm{SNR} > \Theta)$, for some received SNR threshold Θ.

2.2 Buffering Scheme

We consider the following buffering policy *for each flow* in the network, which obeys the following two rules:

1. All the buffering in the network is performed at the source nodes, while each relay node has a buffer size of unity *for each flow it is associated with*. Thus, all the queuing occurs at the source, while relay nodes may hold at most one packet (per flow). We also take that the source nodes are backlogged, i.e., they always have packets to transmit.
2. Incoming transmissions are not accepted by relays if their buffer (corresponding to that flow) already contains a packet.

These rules together mean that a successful transmission may occur only when a node has a packet and its target node's buffer is empty.

Employing this transmission scheme has several benefits such as keeping the in-network packet end-to-end delay small and helping regulate traffic flow in a completely distributed manner. More details on the benefits of this "single packet buffering" policy are provided in our earlier work [13] (and references therein).

3 Preliminaries

We now review the totally asymmetric simple exclusion process, a subfield in statistical mechanics that deals with the flow of particles across a lattice grid and studies their interactions. Later, we will use some results from the TASEP literature to characterize the delay and throughput performances of ad hoc networks.

3.1 An Overview of TASEPs

The TASEP refers to a family of simple stochastic processes used to describe the dynamics of self-driven systems with several interacting particles and is a paradigm for nonequilibrium systems [5]. The classical 1D TASEP model with open boundaries is defined as follows: consider a system with $N+1$ sites, numbered 0 to N. Site 0 is taken to be the source that injects particles into the system. The model is said to have open boundaries, meaning that particles are injected into the system at the left boundary (site 1) and exit the system on the right boundary (site N). The *configuration* of site i, $1 \le i \le N$ at time t is denoted by $\tau_i^{(N)}[t]$ (or simply by $\tau_i[t]$), which can only take values in $\{0, 1\}$, i.e., each site $1 \le i \le N$ may either be *occupied* (denoted as $\tau_i[t] = 1$) or *empty* (denoted as $\tau_i[t] = 0$). The source, however, is taken to be always occupied ($\tau_0[t] \equiv 1, \forall t > 0$). Also, at $t = 0$, all sites other than the source are empty ($\tau_i[0] = 0, 0 < i \leq N$).

In the discrete-time version of the TASEP, the movement of particles is defined to occur in time steps. Specifically, let $(\tau_1[t], \tau_2[t], \ldots, \tau_N[t]) \in \{0,1\}^N$ denote the configuration of the system in time slot t. In the subsequent time slot $t+1$, a set of sites is chosen at first, depending on the *updating procedure*. Then, for every site chosen, if it contains a particle and the neighboring site on its right has none, then the particle hops from that site to its neighbor with a certain probability (which in general, is site dependent).

In this paper, we consider TASEPs with the *random sequential update* wherein a single site is uniformly randomly picked (with probability (w.p.) $1/(N+1)$) for transmission in each time step, and particle hopping is performed as per the aforementioned rules. Formally, supposing that the ith site is picked in time slot t. Then, if $1 \leq i \leq N-1$, the particle on site i (if there is any) jumps to site $i+1$ (provided it is empty) w.p. p, i.e.,

$$\mathbb{P}(\tau_i[t+1] = 0) = 1 - \tau_i[t](1 - p + p\tau_{i+1}[t])$$
$$\mathbb{P}(\tau_i[t+1] = 1) = \tau_i[t](1 - p + p\tau_{i+1}[t]),$$

and

$$\mathbb{P}(\tau_{i+1}[t+1] = 0) = (1 - \tau_{i+1}[t])(1 - p\tau_i[t])$$
$$\mathbb{P}(\tau_{i+1}[t+1] = 1) = p\tau_i[t] + \tau_{i+1}[t](1 - p\tau_i[t]).$$

If $i = 0$, site 1 remains occupied at time $t+1$ if it was occupied at time t and gets occupied w.p. αp if it was empty. Accordingly,

$$\mathbb{P}(\tau_1[t+1] = 0) = (1 - \alpha p)(1 - \tau_1[t])$$
$$\mathbb{P}(\tau_1[t+1] = 1) = \alpha p + (1 - \alpha p)\tau_1[t].$$

If $i = N$, site N remains empty at $t+1$ if it was empty at time t, and gets emptied w.p. βp if it was occupied. Thus,

$$\mathbb{P}(\tau_N[t+1] = 0) = 1 - (1 - \beta p)\tau_N[t]$$
$$\mathbb{P}(\tau_N[t+1] = 1) = (1 - \beta p)\tau_N[t].$$

In this manner, the particles are transported from site 0 through the system until their eventual exit at site N. The quantities α, β, and p may be regarded as the "influx," and "outflux" rates and the hopping probability, respectively.

It is apparent from the description of the TASEP model that it exhibits a similarity to a flow in an ad hoc network. The sites can be taken to represent the relay nodes and the particles of the packets. The hopping probability p is analogous to the link reliability p_s while the exclusion principle models the unit buffer size at the relay nodes. Also, the random sequential update relates to the r-TDMA MAC scheme, and the condition $\tau_0[t] = 1, \forall t$, models the fact that the source node is always backlogged. Fig. 1 depicts the TASEP-equivalent network flow, wherein we assume that the source has a large buffer and regulates the packet flow into a TASEP model.

3.2 The Matrix Product Ansatz Formulation

The starting point for studying the stochastic 1D TASEP model is to write down its *master equation*. Let $P(\tau, t)$ denote

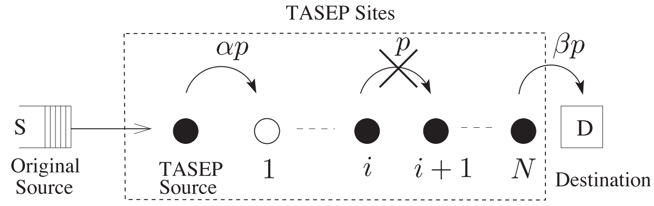

Fig. 1. TASEP-equivalent network flow along with the (site dependent) hopping probabilities αp, βp, and p. The (backlogged) source node with a large buffer connected to the TASEP particle flow model with $N+1$ sites, each with a buffer size of unity. Filled circles indicate occupied sites, and the others indicate holes. Jumping from site j to k is possible only if the configuration (τ_j, τ_k) is (1, 0). In the above example, hopping is not possible between sites i and $i+1$.

the probability of finding the system in the configuration $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$ in time slot t. The master equation describes the evolution of the system with time and takes the form

$$\Delta(P(\tau, t)) = \sum_{\tau'} \left[\xi(\tau', \tau)P(\tau', t) - \xi(\tau, \tau')P(\tau, t)\right],$$

where $\Delta(P(\tau, t)) = P(\tau, t+1) - P(\tau, t)$, and $\xi(\tau, \tau')$ denotes the rate of transition from τ to another configuration τ'. For further details on the master equation and its formulation, we refer the reader to [5], [16].

Interestingly, in the long time limit ($t \to \infty$), the probability of finding the system in any configuration τ becomes independent of t, i.e., $\lim_{t\to\infty} \Delta(P(\tau, t)) = 0$ [5]. The TASEP flow is then said to have reached a *steady state*. Solving for the steady-state configuration probabilities is a formidable task which may be accomplished by considering recursion-based techniques on the system size (see for, e.g., [14], [15]). A more elegant and direct procedure, however, is to use a matrix product ansatz [16], wherein the probability of each configuration at steady state is decomposed into a product of matrices.

According to the MPA formulation [16], the probability of finding the TASEP system in the configuration $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$ at steady state is independent of t and given by

$$P(\tau) = \frac{\langle W| \prod_{i=1}^{N} (\tau_i D + (1 - \tau_i)E)|V\rangle}{\langle W|C^N|V\rangle}, \tag{1}$$

where D and E are square matrices that operate on occupied and empty sites, respectively, $C = D + E$, and $|V\rangle$ and $\langle W|$ are column and row vectors, respectively (represented here by the "ket" and "bra" notation). In general, the matrices D, E and vectors V, W in (1) are all infinite dimensional.[1] A convenient choice of the matrices and vectors (assuming $p > 0$) is [5]

$$D = \frac{1}{p}\begin{pmatrix} 1/\beta & \gamma_1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

1. The only case for which the matrices are finite dimensional (in fact, scalars) is when $\alpha + \beta = 1$ [16].

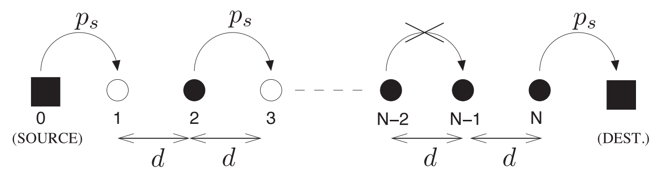

Fig. 2. A regular wireless line network. The backlogged source (node 0) attempts to deliver packets to the destination via N relays, each with unit-sized buffer. The hopping probability across each node is p_s.

$$E = \frac{1}{p}\begin{pmatrix} (1-\alpha p)/\alpha & 0 & 0 & 0 & \cdots \\ \gamma_2 & 1-p & 0 & 0 & \cdots \\ 0 & 1-p & 1-p & 0 & \cdots \\ 0 & 0 & 1-p & 1-p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad (2)$$

with

$$\langle W| = (1,0,0,\ldots) \quad \text{and} \quad |V\rangle = (1,0,0,\ldots)^T,$$

where $(\cdot)^T$ denotes transpose. Here, γ_1 and γ_2 may be chosen so as to satisfy

$$\gamma_1\gamma_2 = \frac{1}{\alpha\beta p}[1-p-(1-\alpha p)(1-\beta p)].$$

In conclusion, the MPA provides an analytical framework for describing the asymmetric exclusion process in a completely algebraic manner. We will employ it extensively for our analysis, in particular in the next section.

4 Throughput and Delay Analyses of a Wireless Line Network

We now use some existing results from the random sequential TASEP literature to study wireless networks. As a first step in this direction, we consider a simple linear network model running the r-TDMA MAC scheme and evaluate the steady-state throughput and the average end-to-end delay for a packet. Additionally, we use the MPA framework extensively to characterize the delay pmf across each hop in closed form and measure the correlations between the delays experienced by packets across adjacent hops along the flow.

The line network model considered comprises a single source node S intending to deliver packets to a destination node D in a multihop fashion via N relay nodes (see Fig. 2). We take the arrangement of nodes to be a regular lattice with equal separation between adjacent nodes d. The attenuation in the channel is modeled as the product of a Rayleigh fading component and a large scale path loss component with exponent γ. Since the fading power is exponentially distributed, we obtain

$$p_s = \Pr[\mathrm{SNR} > \Theta] = \exp(-\Theta N_0 d^\gamma), \qquad (3)$$

for each link in the system. This is equivalent to taking $p = p_s$ in the corresponding TASEP model. We also choose the (analytically tractable) operating point $\alpha = \beta = 1$ for which the network accepts as many packets as it can (when the first relay node's buffer is empty), and also provides the highest possible service rate.[2]

When $\alpha = \beta = 1$ and $p = p_s$, we may take $\gamma_1 = 1$, and $\gamma_2 = 1 - p_s$ in (2) so that

$$D = \frac{1}{p_s}\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad E = \frac{1-p_s}{p_s} D^T.$$

For these forms of matrices D and E, and vectors W and V, the following two properties hold:

$$C = D + E = p_s DE \qquad (4a)$$

$$p_s^N \langle W|C^N|V\rangle := \eta(N) = \frac{(2N+2)!}{(N+2)!(N+1)!}. \qquad (4b)$$

While (4a) is straightforward to establish, (4b) is a consequence of [16, (80), (81)]. We use these results in the remainder of this section.

4.1 Steady-State Probabilities and Occupancies

Using (1) along with the forms of matrices and vectors discussed earlier, the steady-state probabilities can be computed in a straightforward manner, in particular for small values of N. As examples, we have for $N = 1$,

$$P(0) = \frac{\langle W|E|V\rangle}{\langle W|C|V\rangle} = 1/2, \text{ and } P(1) = \frac{\langle W|D|V\rangle}{\langle W|C|V\rangle} = 1/2.$$

Likewise, one can show for $N = 2$,

$$P(0,0) = P(0,1) = P(1,1) = 1/5, \text{ and } P(1,0) = 2/5.$$

Next, we compute the steady-state *occupancy* of each node $0 \le i \le N$, defined as the probability that it is occupied at steady state. Hereafter, we use the notation τ_i to denote the configuration of node i, $0 \le i \le N$ at steady state. From (1), we obtain the occupancy of node i to be

$$\mathbb{P}(\tau_i = 1) = \frac{\langle W|C^{i-1}DC^{N-i}|V\rangle}{\langle W|C^N|V\rangle}, \quad 0 \le i \le N.$$

From [14, (48)], this simplifies to

$$\mathbb{E}\tau_i = \frac{1}{2} + \frac{1}{4}\frac{(2i)!}{(i!)^2}\frac{(N!)^2}{(2N+1)!}\frac{(2N-2i+2)!}{[(N-i+1)!]^2}(N-2i+1). \qquad (5)$$

Note that since τ_i can take values only in $\{0,1\}$, $\mathbb{P}(\tau_i = 1) = \mathbb{E}\tau_i$ and $\mathbb{P}(\tau_i = 0) = 1 - \mathbb{E}\tau_i$.

Surprisingly, the node occupancies are independent of p_s. Also, notice the particle-hole symmetry,[3] i.e., $\mathbb{E}\tau_i = 1 - \mathbb{E}\tau_{N+1-i}$. Thus, $\sum_{i=0}^N \mathbb{E}\tau_i = 1 + N/2$. In a system with an odd number of relays, the middle relay has an occupancy of exactly $1/2$. The steady-state occupancies for an r-TDMA-based flow along $N = 5$ relays in depicted in Fig. 3.

4.2 Steady-State Throughput

We now derive the *throughput* of the r-TDMA-based line network at steady state, defined as the average number of

2. Equivalently, the rate of packet flow across the r-TDMA-based network is maximized when $\alpha = \beta = 1$.

3. The movement of particles (packets) to the right is equivalent to the movement of *holes* (nodes with empty buffers) to the left.

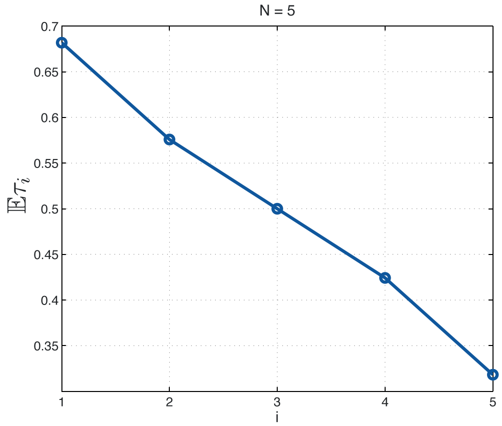

Fig. 3. The steady-state occupancy of nodes in an r-TDMA-based flow along $N = 5$ relays. Notice the particle-hole symmetry, i.e., $\mathbb{E}\tau_i = 1 - \mathbb{E}\tau_{N+1-i}$.

packets successfully delivered (to the destination) in a unit step of time. We have the following result.

Theorem 4.1. *For the r-TDMA-based line network with N nodes, the throughput at steady state is*

$$T = \frac{p_s(N+2)}{2(N+1)(2N+1)}. \tag{6}$$

Proof. At any instant of time, node N's buffer contains a packet w.p. τ_N; furthermore, it is picked for transmission w.p. $1/(N+1)$, and the transmission is successful w.p. p_s. Thus, the throughput of the line network is simply

$$T = p_s\mathbb{E}\tau_N/(N+1). \tag{7}$$

Using (5) in (7), we obtain the desired result. □

Since the network reliability is 100 percent, the throughput across each link is the same as specified by (7). Noting that the probability that node i has a packet and node $i+1$ none is $\mathbb{P}(\tau_i = 1, \tau_{i+1} = 1) = \mathbb{E}[\tau_i(1-\tau_{i+1})]$, T may also be obtained using any of the $N+1$ equivalent expressions

$$T = p_s\mathbb{E}[\tau_i(1-\tau_{i+1})]/(N+1), \tag{8}$$

for any $i \in [0, N]$. The system throughput at steady state is proportional to the link reliability and upper bounded by $p_s/4$, but decreases with increasing system size.

Note that instead of picking any of the $N+1$ nodes randomly, if one only chooses among the nodes having a packet (as in CSMA), the throughput is improved by a factor of $N + 1/(\sum_{i=0}^{N}\mathbb{E}\tau_i) = 2(N+1)/(N+2)$, i.e., $T = p_s/(2N+1)$.

4.3 Average End-to-End Delay at Steady State

In this paper, we are also interested in the *in-network delay*[4] measured by the number of time slots for by the packet at the head of the source's queue to be delivered (to the destination). We first evaluate the average end-to-end delay incurred by packets along each node in the network. Later, we derive the complete distribution of the delays.

4. There is no queueing delay at the source node since it is considered to be always backlogged.

Corollary 4.2. *For the wireless multihop network with N relays running the r-TDMA scheme, the average delay experienced by a packet at node i is*

$$\mathbb{E}D_i = \frac{2(N+1)(2N+1)\mathbb{E}\tau_i}{(N+2)p_s}, \quad 0 \le i \le N. \tag{9}$$

Consequently, the average end-to-end delay is

$$\mathbb{E}D_{\mathrm{e2e}} = \sum_{i=0}^{N}\mathbb{E}D_i = \frac{2N^2+3N+1}{p_s}. \tag{10}$$

Proof. Recall that the rate of packet flow across each node is equal to T, and that the average number of packets at node i, $0 \le i \le N$ is $\mathbb{E}\tau_i$. From Little's theorem [6], the average delay at node i is simply $\mathbb{E}\tau_i/T$. □

We see that the average end-to-end delay is proportional to the node occupancies and inversely proportional to the link reliability. Also, it is interesting to note that the product of throughput and delay is $1 + N/2$, which is independent of p_s.

4.4 Delay Distributions

In this section, we analytically derive the pmfs of the steady-state delays incurred by packets at each node along the linear flow, i.e., we evaluate $\mathbb{P}(D_i = k)$, $k \ge 1$, $0 \le i \le N$ in closed form.

To this end, suppose that a packet arrives at a node i in an arbitrary time slot t (at steady state). The three events that need to occur in the following order for the packet to be able to hop to node $i+1$ successfully are:

1. Node $i+1$ has an empty buffer.
2. Node i is picked for transmission.
3. Node i's transmission is successful.

While (2) occurs w.p. $1/(N+1)$, (3) happens (independently of (2)) w.p. p_s. Thus, at time t, if node $i+1$'s buffer is empty, the delay experienced by the packet at node i is simply geometrically distributed with mean $(N+1)/p_s$, i.e.,

$$\mathbb{P}(D_i = k) = \frac{p_s}{N+1}\left(1 - \frac{p_s}{N+1}\right)^{k-1}.$$

If instead, there is another packet present in node $i+1$'s buffer, however, no packet at node $i+2$'s buffer, the probability that the delay at the ith node is k time slots is equal to the probability that a single successful transmission (of the packet at node $i+1$) occurs within $k-1$ slots, and then the packet at node i hops in the kth time slot. Extending this argument, if j nodes adjacent to node i have a packet, and the $j+1$th adjacent node has none, i.e., if

$$(\tau_{i+1}, \ldots, \tau_{i+j}, \tau_{i+j+1}) = (\underbrace{1, \ldots, 1}_{j \text{ ones}}, 0), \quad j \ge 0,$$

then the packet at node i will successfully hop to node $i+1$ in exactly k time steps if j packets (those at nodes $i+j, i+j-1, \ldots, i+1$ in that order) hop within $k-1$ time slots, and then, the packet present at node i hops (in the kth time slot).

Let $e_{i,j}$ denote the event that given a packet arrives at node i (at some time t), j nodes adjacent to it are occupied. We now compute $\Delta_{i,j}^{(N)} := \mathbb{P}(e_{i,j})$. We have

$$\begin{aligned}
&\mathbb{P}\left(\tau_{i+1}[t] = 1, \ldots, \tau_{i+j}[t] = 1, \tau_{i+j+1}[t] = 0 \mid {}^{\text{packet arrives}}_{\text{at node } i}\right) \\
&= \frac{\mathbb{P}\left(\tau_{i+1}[t] = 1, \ldots, \tau_{i+j}[t] = 1, \tau_{i+j+1}[t] = 0, {}^{\text{packet arrives}}_{\text{at node } i}\right)}{\mathbb{P}\text{ (packet arrives at node } i)} \\
&= P(\tau_{i-1}[t-1] = 1, \tau_i[t-1] = 0, \tau_{i+1}[t-1] \\
&= 1, \ldots, \tau_{i+j}[t-1] = 1, \tau_{i+j+1}[t-1] = 0) \\
&\times \frac{1}{P(\tau_{i-1}[t-1] = 1, \tau_i[t-1] = 0)} \\
&\times \frac{\mathbb{P}(\text{the packet at node } i-1 \text{ hops to node } i)}{\mathbb{P}(\text{the packet at node } i-1 \text{ hops to node } i)}.
\end{aligned}$$

Using the MPA formalism, we may write at steady state,

$$\begin{aligned}
\Delta_{i,j}^{(N)} &= \frac{\langle W|C^{i-2}DED^{j}EC^{N-i-j-1}|V\rangle}{\langle W|C^{i-2}DEC^{N-i}|V\rangle} \\
&\overset{(a)}{=} \frac{\langle W|C^{i-1}D^{j-1}C^{N-i-j}|V\rangle}{p_s\langle W|C^{N-1}|V\rangle},
\end{aligned} \tag{11}$$

where in (a), we have used (4a) thrice (twice in the numerator term and once in the denominator term).

The evaluation of $\Delta_{i,j}^{(N)}$ is relatively straightforward for small values of j. For instance,

$$\begin{aligned}
\Delta_{i,0}^{(N)} &= \frac{\langle W|C^{i-2}DEEC^{N-i-1}|V\rangle}{\langle W|C^{i-2}DEC^{N-i}|V\rangle} \\
&= \frac{\langle W|C^{i-1}EC^{N-i-1}|V\rangle}{\langle W|C^{N-1}|V\rangle} \\
&= 1 - \mathbb{E}\tau_i^{(N-1)},
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\Delta_{i,1}^{(N)} &= \frac{\langle W|C^{i-2}DEDEC^{N-i-2}|V\rangle}{\langle W|C^{i-2}DEC^{N-i}|V\rangle} \\
&= \frac{\langle W|C^{N-2}|V\rangle}{\langle W|C^{N-1}|V\rangle} \\
&= \eta(N-2)/\eta(N-1).
\end{aligned} \tag{13}$$

In order to compute $\Delta_{i,j}^{(N)}$ for higher values of j, we use the following lemmas.

Lemma 4.3. *The following relationship holds for $j \geq 2$:*

$$\Delta_{i,j}^{(N)} = \Delta_{i,j-1}^{(N)} - \Delta_{i,j-2}^{(N-1)}\eta(N-2)/\eta(N-1). \tag{14}$$

The proof is presented in the appendix, which can be found on the Computer Society Digital Library at http://doi.ieeecomputersociety.org/10.1109/TMC.2011.96.

Lemma 4.4. *For $j \geq 2$, we have*

$$\begin{aligned}
\Delta_{i,j}^{(N)} &= \sum_{k=0}^{\lfloor\frac{j-1}{2}\rfloor}(-1)^k\,\frac{\eta(N-k-2)}{\eta(N-1)} \\
&\times\left[\binom{j-k-2}{k}\mathbb{E}\tau_i^{(N-k-2)} + \binom{j-k-2}{k-1}\right],
\end{aligned} \tag{15}$$

where $\mathbb{E}\tau_i^{(N)}$ denotes the occupancy of node i in the flow with N relays.

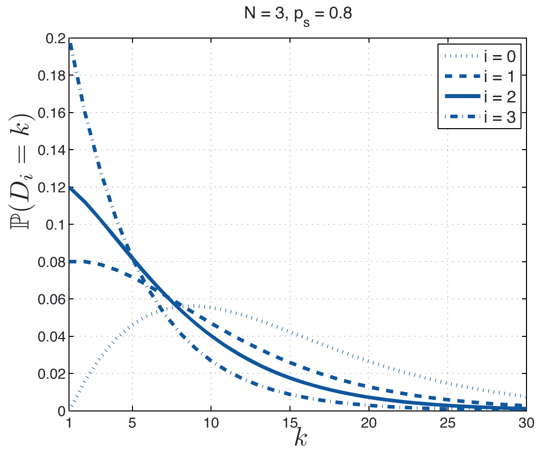

Fig. 4. The pmf of the delay incurred by packets at various nodes in the line network with $N = 3$. For this plot, all link reliabilities are taken to be equal to $p_s = 0.8$.

The proof is presented in the appendix, available in the online supplemental material.

Theorem 4.5. *The pmf of the packet delay at node i, $0 \leq i \leq N$ is given by*

$$\mathbb{P}(D_i = k) = \sum_{j=0}^{N-i}\Delta_{i,j}^{(N)}\binom{k-1}{j}\xi^{j+1}(1-\xi)^{k-1-j}, \tag{16}$$

where $\xi = p_s/(N+1)$.

Proof. The conditional probability $\mathbb{P}(D_i = k \mid e_{i,j})$ is the probability that j packets (present at nodes $i+1, \ldots i+j$) hop out successfully in $k-1$ time slots, and then, the packet at node i is transmitted successfully only in the kth time slot. Hence,

$$\mathbb{P}(D_i = k \mid e_{i,j}) = \binom{k-1}{j}\xi^j(1-\xi)^{k-1-j} \times \xi.$$

Summing up the joint pmf $\mathbb{P}(D_i = k, e_{i,j})$ over all the possible values of j $(0 \leq j \leq N-i)$ yields the desired result, i.e.,

$$\mathbb{P}(D_i = k) = \sum_{j=0}^{N-i}\mathbb{P}(e_{i,j})\mathbb{P}(D_i = k \mid e_{i,j}), \quad k > 0,$$

which is equivalent to (16). □

Fig. 4 plots the delay pmfs at each node in a line network with $N = 3$. Note that apart from the delay at the final relay, none of the other delays are geometrically distributed, i.e., they are not memoryless. Also note that $D_0 \geq 2$. This may be explained by the fact that whenever a packet hops out of the source node (node 0), another packet arrives at the head of the source. Thus, the packet at node 0 has to wait for at least one time slot (for the packet at node 1 to hop out) before attempting to hop.

4.5 Joint Delay Distributions

Since the flow of packets in a wireless multihop network is relayed across multiple links, the delays experienced by a packet across hops are correlated. As mentioned earlier, the study of delay correlations has often been neglected in prior work; it is, however, crucial for the design of smarter retransmission and flow control algorithms.

For instance, suppose it is known that the conditional delay probability $\mathbb{P}(D_{i+1} = j \mid D_i = k)$ is high for some specific value $j = \ell$, i.e., given that a packet stayed at node i for k slots, it is likely to be present in node $i+1$'s buffer for ℓ slots. Node $i+1$ can then decide to hold back the transmission of a packet for $\ell - 1$ slots, thus reducing the number of unnecessary transmissions. Knowing the spatial delay correlations also helps determine the variance of the end-to-end delay.

We begin by stating the following simple lemma.

Lemma 4.6. *In a multihop wireless network with N nodes, D_N is independent of all the other hop delays. As a special case, when $N = 1$, D_0 and D_1 are independent.*

Proof. Irrespective of the delay experienced by a packet at any arbitrary node, it can hop from node N to the destination (node $N+1$) if node N is picked, and its transmission is successful. Thus, D_N follows a geometric distribution with mean $(N+1)/p_s$ and is independent of all other delays. □

We next compute $\mathbb{P}(D_{i+1} = \ell, D_i = k)$, i.e., the probability that a packet will stay at nodes i and $i+1$ for k and ℓ slots, respectively, at steady state? The same procedure may be extended (with extra care) to evaluate the joint pmfs of the delays at nodes farther apart.

To this end, let e_{i,j_1,j_2} denote the event that given a packet arrives at node i, we have

$$\begin{aligned}&(\tau_{i+1}, \ldots, \tau_{i+j_1}, \tau_{i+j_1+1}, \tau_{i+j_1+2} \ldots, \tau_{i+j_1+j_2+1}, \tau_{i+j_1+j_2+2})\\ &= \Bigg(\underbrace{1, \ldots, 1}_{j_1 \text{ ones}}, 0, \underbrace{1, \ldots, 1}_{j_2 \text{ ones}}, 0\Bigg).\end{aligned}$$

We first evaluate $\kappa_{i,j_1,j_2}^{(N)} := \mathbb{P}(e_{i,j_1,j_2})$. Using the same idea as in (11), we may write

$$\begin{aligned}\kappa_{i,j_1,j_2}^{(N)} &= \frac{\langle W|C^{i-2}DED^{j_1}ED^{j_2}EC^{N-i-j_1-j_2-2}|V\rangle}{\langle W|C^{i-2}DEC^{N-i}|V\rangle}\\ &= \frac{\langle W|C^{i-1}D^{j_1}ED^{j_2-1}C^{N-i-j_1-j_2-1}|V\rangle}{p_s\langle W|C^{N-1}|V\rangle}.\end{aligned}$$

Simplifying further, we get

$$\begin{aligned}\kappa_{i,j_1,j_2}^{(N)} &\overset{(a)}{=} \frac{\langle W|C^{i-1}D^{j_1-1}CD^{j_2-1}C^{N-i-j_1-j_2-1}|V\rangle}{p_s^2\langle W|C^{N-1}|V\rangle}\\ &\overset{(b)}{=} \frac{\eta(N-2)}{\eta(N-1)}\Bigg[\frac{\langle W|C^{i-1}D^{j_1+j_2-1}C^{N-i-j_1-j_2-1}|V\rangle}{p_s\langle W|C^{N-2}|V\rangle}\\ &\quad + \frac{\langle W|C^{i-1}D^{j_1-1}ED^{j_2-1}C^{N-i-j_1-j_2-1}|V\rangle}{p_s\langle W|C^{N-2}|V\rangle}\Bigg]\\ &= \frac{\eta(N-2)}{\eta(N-1)}\Big[\Delta_{i,j_1+j_2}^{(N-1)} + \kappa_{i,j_1-1,j_2}^{(N-1)}\Big].\end{aligned} \tag{17}$$

To derive (a), we used (4a) for the term in the numerator, and to derive (b), we used the identity $C = D + E$.

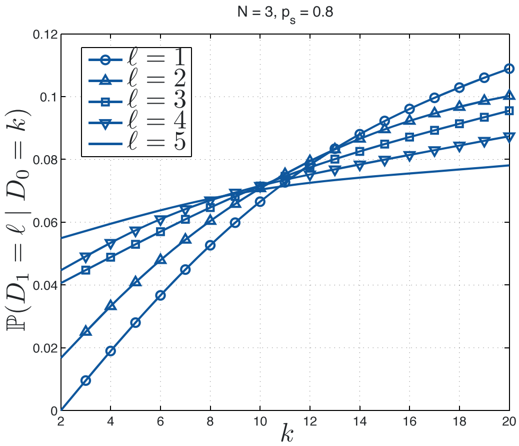

Fig. 5. The conditional delay pmfs $\mathbb{P}(D_1 = \ell \mid D_0 = k)$ for several values of ℓ and k.

We obtain closed-form expressions for $\kappa_{i,j_1,j_2}^{(N)}$ considering the following two cases:

Case 1: $j_1 = 0$. From (17), we obtain

$$\begin{aligned}\kappa_{i,0,j_2}^{(N)} &= \frac{\langle W|C^{i-1}ED^{j_2}EC^{N-i-j_2-1}|V\rangle}{\langle W|C^{N-1}|V\rangle}\\ &= \frac{\langle W|C^{i}D^{j_2}EC^{N-i-j_2-1}|V\rangle}{p_s\langle W|C^{N-1}|V\rangle}\\ &\quad - \frac{\langle W|C^{i-1}D^{j_2+1}EC^{N-i-j_2-1}|V\rangle}{p_s\langle W|C^{N-1}|V\rangle}\\ &= \Delta_{i+1,j_2}^{(N)} - \Delta_{i,j_2+1}^{(N)}.\end{aligned} \tag{18}$$

Case 2: $j_1 > 0$. Using the recursive equation (17), we obtain

$$\kappa_{i,j_1,j_2}^{(N)} = \frac{\eta(N-j_1-1)}{\eta(N-1)}\left(\Delta_{i+1,j_1+j_2}^{(N-j_1)} + \sum_{s=2}^{j_1}\Delta_{i,s+j_2}^{(N-j_1-1+s)}\right).$$

The following theorem establishes the joint pmfs between delays across adjacent hops in the network.

Theorem 4.7. *The joint pmf of the delays at nodes i and $i+1$ is given by*

$$\begin{aligned}\mathbb{P}(D_{i+1} = \ell, D_i = k) &= \sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2}\sum_{j=j_1}^{2j_1+j_2}\kappa_{i,j_1,j_2}^{(N)}\binom{k-1}{j}\\ &\times\binom{\ell-1}{2j_1+j_2-j}\xi^{2j_1+j_2+2}(1-\xi)^{k+\ell-2-2j_1-j_2}\\ &+\sum_{j_1=0}^{s_3}\sum_{j=j_1}^{2j_1-1}\kappa_{i,j_1,j_2}^{(N)}\binom{k-1}{j}\binom{\ell-1}{2j_1-1-j}\xi^{2j_1+1}\\ &\times(1-\xi)^{k+\ell-1-2j_1},\end{aligned} \tag{19}$$

where $\xi = p_s/(N+1)$, $s_1 = \min\{k-1, N-i-1\}$, $s_2 = \min\{k+\ell-2j_1-2, N-i-1-j_1\}$, and $s_3 = \min\{k-1, (k+\ell-1)/2\}$.

The proof is presented in the appendix, available in the online supplemental material.

The conditional delay pmf may be obtained by using (19) together with (16). Fig. 5 plots the conditional delay pmfs $\mathbb{P}(D_1 = \ell \mid D_0 = k)$ in a line network with $N = 3$ and $p_s = 0.8$, for several values of ℓ and k.

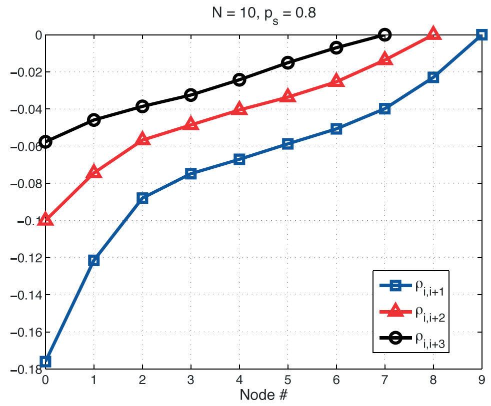

Fig. 6. The correlation coefficients $\rho_{i,i+1}$, $\rho_{i,i+2}$, and $\rho_{i,i+3}$ for $p_s = 0.8$ in a multihop r-TDMA-based system with $N = 10$ relays. The delay correlations across nodes farther apart and closer to the destination are seen to be relatively light.

4.6 Empirical Results

Evaluating the joint pmfs between delays across nodes lying farther apart can be performed by essentially following the aforementioned procedure, but it gets computationally intensive and unwieldy. Instead, we resort to simulation and present the behavior of the *spatial delay correlation coefficients*. The correlation coefficient between delays at nodes i and j is defined as

$$\rho_{i,j} = \frac{\mathbb{E}\left[(D_i - \mathbb{E}D_i)(D_j - \mathbb{E}D_j)\right]}{\sigma_{D_i}\sigma_{D_j}},$$

where σ_{D_i} and σ_{D_j} represent the standard deviations of the delays at node i and j, respectively.

Fig. 6 plots the empirical values of correlation coefficients across one-, two- and three-hop neighbors in an r-TDMA-based wireless network with $N = 10$ relays and $p_s = 0.8$. Observe that all the delay correlation coefficients are nonpositive. This can be explained by noting that if the transmission of a packet is delayed at any node, the adjacent nodes' buffers get emptied so that the packet traverses faster across them. Likewise, if the waiting time of a packet at any particular node is small, the neighboring relay node buffers are still occupied and therefore it takes longer for the packet to get transported across the system. Also, delays across hops closer to the destination, and delays across nodes farther apart are relatively lightly correlated compared to the corresponding values near the source node. In fact, $\forall i$, $\rho_{i,N} = 0$, since D_N is independent of all other delays (which is also a consequence of Lemma 4.6).

5 More Complex Topologies

So far, we have only considered the linear wireless network model. In this section, we consider more complex ad hoc networks comprising intersecting routes, i.e., networks consisting of flows that travel through common relays. We also propose the partial mean-field approximation, a statistical mechanics-based tool which may be used to approximate the throughput performance of networks with arbitrary topologies.

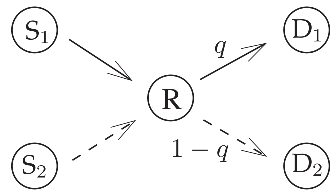

Fig. 7. The two flows $\mathrm{S}_1 \to \mathrm{R} \to \mathrm{D}_1$ and $\mathrm{S}_2 \to \mathrm{R} \to \mathrm{D}_2$, each occurring via the relay node R are represented by solid and dashed arrows, respectively. When the relay node contains two packets, it routes either the packet meant for D_1 w.p. q or the one for D_2 w.p. $1-q$. The probability of a successful transmission across all links is p_s.

5.1 Two Two-Hop Flows via a Common Relay

We begin by considering the network model depicted in Fig. 7. It comprises two source nodes S_1 and S_2 (each numbered 0 with respect to (w.r.t) its corresponding flow) intending to deliver packets to destinations D_1 and D_2 (each numbered 2), respectively, each via a common relay node R (numbered 1). Here, we take that the relay node has a buffer size of two since it accommodates two flows. Furthermore, the r-TDMA dictates that in any time slot, only one of the three nodes (S_1, S_2, or R) is (uniformly) randomly picked (w.p. $1/3$) for transmission. Let p_s denote the reliability of each link. Whenever R is picked, any one of the following event occurs:

1. If the relay's buffer has no packet, it obviously does not transmit anything.
2. If the relay's buffer contains only one packet (intended for either of the destinations), that packet is transmitted.
3. If the relay's buffer has two packets (to be forwarded to both the destinations), it transmits either the packet intended for D_1 w.p. q or the packet meant for D_2 w.p. $1-q$. Note that priority-based routing may be modeled by setting $q = 1$ (prioritizing the first flow) or $q = 0$ (for the second flow). $q = 0.5$ models having equal priorities for the flows.

For notational convenience, let $\tau_j^{[i]}$ represent the steady-state configurations for the buffers across the two flows, $i = \{1, 2\}$, for each of the three nodes involved in each flow, numbered $j = \{0, 1, 2\}$. By definition, $\tau_0^{[1]} = \tau_0^{[2]} = 1$ and $\tau_2^{[1]} = \tau_2^{[2]} = 0$. We shall now derive the steady-state throughput, $T^{[1]}$, for the first flow; $T^{[2]}$ may simply be obtained by replacing q by $1-q$.

Using the fact that for each flow, the throughput across each link is the same (8), we get from 1-3,

$$\mathbb{E}\left[1 - \tau_1^{[1]}\right] = \mathbb{E}\left[\tau_1^{[1]}\left(1 - (1-q)\tau_1^{[2]}\right)\right], \tag{20}$$

and

$$\mathbb{E}\left[1 - \tau_1^{[2]}\right] = \mathbb{E}\left[\tau_1^{[2]}\left(1 - q\tau_1^{[1]}\right)\right] \tag{21}$$

for the first and second flows, respectively. In order to solve the above equations analytically, we can use the *mean-field approximation* (MFA) [5], according to which all the correlations between the buffer occupancies are neglected. Mathematically, the MFA takes that

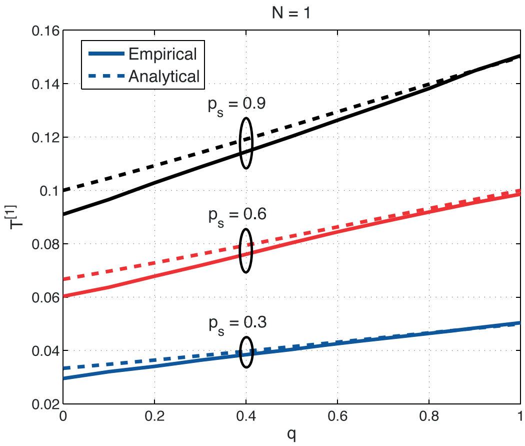

Fig. 8. Steady-state throughput across the first flow, $T^{[1]}$ versus q for several values of the link success probability p_s. The results obtained numerically (dashed lines) closely approximate the empirical results (solid lines).

$$\mathbb{E}\big[\tau_i^{[j]}\tau_k^{[l]}\big] = \mathbb{E}\tau_i^{[j]}\mathbb{E}\tau_k^{[l]},$$

for all (valid) node pairs (i,k) and flow pairs (j,l).

For this example in particular, we assume that $\mathbb{E}[\tau_1^{[1]}\tau_1^{[2]}] = \mathbb{E}\tau_1^{[1]}\mathbb{E}\tau_1^{[2]}$. Employing the MFA and the simplified notation $\mathbb{E}\tau_1^{[1]} = x$, $\mathbb{E}\tau_1^{[2]} = y$ in (20) and (21), we obtain

$$\begin{aligned} 1 - x &= x - (1-q)xy \\ 1 - y &= y - qxy. \end{aligned}$$

Solving the above equations simultaneously, we obtain the only meaningful solution as

$$x = \frac{2q + 3 - \sqrt{4q^2 - 4q + 9}}{4q}.$$

Since the channel access probability for each node in the system is $1/3$, we see that the throughput for the flow $\mathrm{S}_1 \to \mathrm{R} \to \mathrm{D}_1$ is given by

$$T^{[1]}(q) = \frac{p_s\mathbb{E}\tau_1^{[1]}}{3} = \frac{p_s(2q - 3 + \sqrt{4q^2 - 4q + 9})}{12q}. \tag{22}$$

When $q = 1$, i.e., when the first flow is always given priority over the second flow, $T^{[1]}(1) = p_s/6$. On the other hand, when $q = 0$, we use the L' Hôpital rule to see that $T^{[1]}(0) = p_s/9$. When both flows are prioritized equally, $T^{[1]}(q) = T^{[2]}(q) = p_s(\sqrt{2} - 1)/3$. The achievable set of throughput for the first flow, $T^{[1]}$ is plotted in Fig. 8 for different values of p_s. For comparison, we have also shown empirical results, which match the theoretical ones (20) closely, in particular when p_s is small.

5.2 Two Three-Hop Flows via a Common Relay

We next consider the case where again, the two source nodes S_1 and S_2 intend to deliver packets to different destinations D_1 and D_2, respectively. Here, however, we take that in each flow, packets traverse two hops each, one of which is the common relay. Evidently, the common relay may be the node numbered 1 or the node numbered 2 (see Fig. 9). The channel access probability for each node is $1/5$.

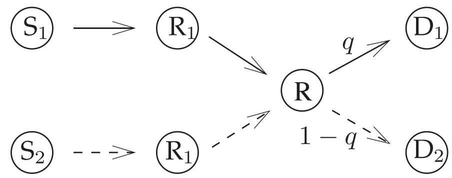

Fig. 9. The two three-hop flows $\mathrm{S}_1 \to \mathrm{R}_1 \to \mathrm{R} \to \mathrm{D}_1$ and $\mathrm{S}_2 \to \mathrm{R}_2 \to \mathrm{R} \to \mathrm{D}_2$, each occurring via the relay R are represented by solid and dashed lines, respectively. In this case, the common relay is the node numbered 2.

5.2.1 Common Relay: Node 1

We first analyze the case wherein the common relay is the node numbered 1. Since the throughput across each link is the same (for each flow), we obtain at steady state,

$$1 - \mathbb{E}\tau_1^{[1]} = \mathbb{E}\big[\tau_1^{[1]}\big(1 - (1-q)\tau_1^{[2]}\big)\big(1 - \tau_2^{[1]}\big)\big] = \mathbb{E}\tau_2^{[1]},$$

and

$$1 - \mathbb{E}\tau_1^{[2]} = \mathbb{E}\big[\tau_1^{[2]}\big(1 - q\tau_1^{[1]}\big)\big(1 - \tau_2^{[2]}\big)\big] = \mathbb{E}\tau_2^{[2]}.$$

Evidently, when $q = 1$, the second flow (the one without the priority) does not affect the throughput across the first flow. Following (5), $\mathbb{E}\tau_2^{[1]} = 2/5; T^{[1]}(1) = p_s\mathbb{E}\tau_2^{[1]}/5 = 0.08p_s$.

For general q, we may use the MFA to analytically evaluate the throughput. Indeed, setting $\mathbb{E}\tau_1^{[1]} = x$, $\mathbb{E}\tau_1^{[2]} = y$, $\mathbb{E}\tau_2^{[1]} = u$, and $\mathbb{E}\tau_2^{[2]} = v$, we obtain the following set of four equations:

$$\begin{aligned} 1 - x &= u, \\ u &= x(1-u)(1-(1-q)y), \\ 1 - y &= v, \\ v &= y(1-v)(1-qx), \end{aligned}$$

which may be solved numerically. It is easily seen that when $q = 0$, the first flow does not affect the throughput across the second flow. From (5), $\mathbb{E}\tau_1^{[2]} = 3/5$, so that

$$\mathbb{E}\tau_1^{[1]} = \frac{9 - \sqrt{65}}{4} \approx 0.234,$$

and $T^{[1]}(0) = \mathbb{E}\tau_1^{[1]}p_s/5 \approx 0.047p_s$.

5.2.2 Common Relay: Node 2

We now consider the case when the common relay is the node numbered 2. For this scenario, we have

$$1 - \mathbb{E}\tau_1^{[1]} = \mathbb{E}\big[\tau_1^{[1]}\big(1 - \tau_2^{[1]}\big)\big] = \mathbb{E}\big[\tau_2^{[1]}\big(1 - (1-q)\tau_2^{[2]}\big)\big],$$

and

$$1 - \mathbb{E}\tau_1^{[2]} = \mathbb{E}\big[\tau_1^{[2]}\big(1 - \tau_2^{[2]}\big)\big] = \mathbb{E}\big[\tau_2^{[2]}\big(1 - q\tau_2^{[1]}\big)\big].$$

Using similar arguments as earlier, we obtain $T^{[1]}(1) = 0.08p_s$. When $q = 0$, $\mathbb{E}\tau_2^{[2]} = 3/5$, and employing the MFA, we have $T^{[1]}(0) = (11 - \sqrt{85})/30 \approx 0.06p_s$.

The above analysis suggests that the throughput across the flow is higher when the bottleneck node is closer to the destination (also see Fig. 10). This is explained by the fact that the node occupancies monotonically decrease with proximity to the destination.

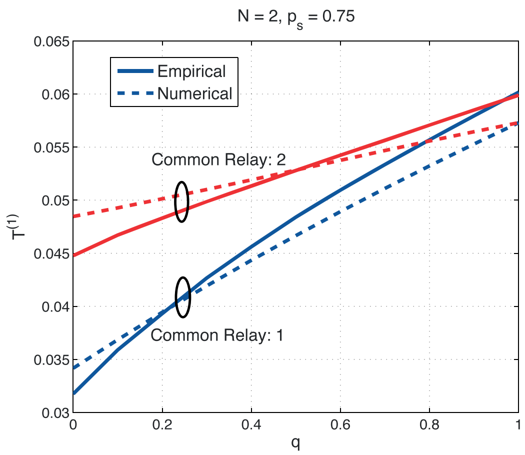

Fig. 10. Steady-state throughput across the first flow, $T^{[1]}$ versus q for $p_s = 0.75$ for different locations of the common relay node. The results obtained numerically (dashed lines) closely approximate the empirical results (solid lines).

5.3 Multiple Flows via a Common Relay

Next, we consider a network topology comprising multiple (>2) flows passing through a common relay (see Fig. 11). Here, the source nodes $\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_n$ attempt to deliver packets to their corresponding destinations $\mathrm{D}_1, \mathrm{D}_2, \ldots, \mathrm{D}_n$, through a common relay node R (that has a buffer size of n). We also take that routing is priority based with packets intended for D_1 having the highest priority and those meant for D_n the lowest. Thus, the relay node transmits the packet meant for node k, $1 \leq k \leq n$, only when it does not have other packets corresponding to the destination nodes D_j, $j < k$ in its buffer.

Since the throughput of each flow is conserved, we obtain the following set of equations:

$$\begin{aligned} 1 - \mathbb{E}\tau_1^{[1]} &= \mathbb{E}\tau_1^{[1]} \\ 1 - \mathbb{E}\tau_1^{[2]} &= \left(1 - \mathbb{E}\tau_1^{[1]}\right)\mathbb{E}\tau_1^{[2]} \\ &\vdots \\ 1 - \mathbb{E}\tau_1^{[n]} &= \prod_{i=1}^{n-1}\left(1 - \mathbb{E}\tau_1^{[i]}\right)\mathbb{E}\tau_1^{[n]}. \end{aligned}$$

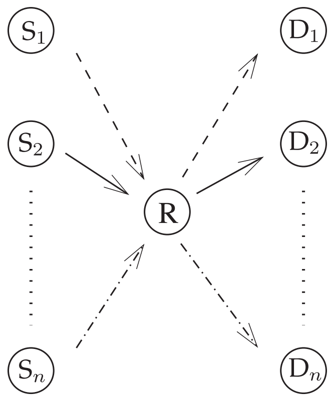

Fig. 11. n flows $\mathrm{S}_1 \to \mathrm{D}_1$, $\mathrm{S}_2 \to \mathrm{D}_2, \ldots, \mathrm{S}_n \to \mathrm{D}_n$ passing through a common relay node R. When routing, packets intended for D_1 are taken to have the highest priority, and those meant for D_n, the lowest.

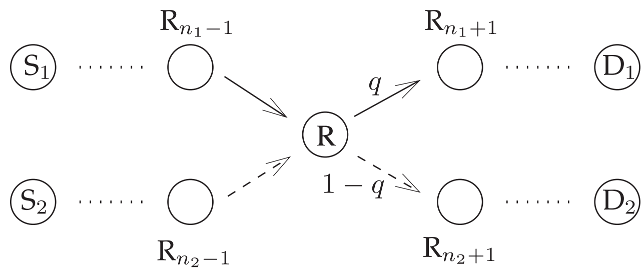

Fig. 12. Two multihop flows $\mathrm{S}_1 \to \mathrm{D}_1$ and $\mathrm{S}_2 \to \mathrm{D}_2$ across N_1 and N_2 nodes each occur via a common relay node R. The common relay is numbered n_1 and n_2 w.r.t. the first and second flows, respectively.

Solving the above set of equations using the MFA yields $\mathbb{E}\tau_1^{[k]} = k/(k+1)$, $1 \leq k \leq n$. In this case, the channel access probability for each node is $1/(n+1)$, so that

$$T^{[k]} = \frac{1 - \mathbb{E}\tau_1^{[k]}}{n+1} = \frac{1}{(k+1)(n+1)}, \quad 1 \leq k \leq n. \tag{23}$$

5.4 The Partial Mean-Field Approximation

While the MFA tightly approximates the throughput performance of networks comprising short flows, it can get loose, in particular when the flows in the network traverse several nodes, since it neglects the correlations between all the node occupancies. In this section, we present the partial mean-field approximation, which (as we shall see later) is more accurate than the MFA. Later, in Section 5.5, we illustrate (via a simple example) how to employ the PMFA framework to evaluate the throughput performance of a network with an arbitrary topology.

We begin by considering a scenario where two general multihop flows (of arbitrary lengths) both pass through a common relay node. Suppose that source node S_1 delivers data to D_1 in a multihop fashion via N_1 nodes, while S_2 forward packets to D_2 via N_2 relays, each via a common relay node R (see Fig. 12). We take that R is numbered $1 \leq n_1 \leq N_1$ w.r.t. the first flow, and $1 \leq n_2 \leq N_2$ w.r.t. the second flow.

In principle, the MFA may be used to compute the steady-state throughput of each flow. Indeed, we get for the first flow

$$\begin{aligned} 1 - \mathbb{E}\tau_1^{[1]} &= \mathbb{E}\left[\tau_1^{[1]}\left(1 - \tau_2^{[1]}\right)\right] = \cdots \\ &= \mathbb{E}\left[\tau_{n_1}^{[1]}\left(1 - \tau_{n_1+1}^{[1]}\right)\left(1 - (1-q)\tau_{n_2}^{[2]}\right)\right] \\ &= \cdots = \mathbb{E}\left[\tau_{N_1-1}^{[1]}\left(1 - \tau_{N_1}^{[1]}\right)\right] = \mathbb{E}\tau_{N_1}^{[1]}, \end{aligned}$$

and the second flow

$$\begin{aligned} 1 - \mathbb{E}\tau_1^{[2]} &= \mathbb{E}\left[\tau_1^{[2]}\left(1 - \tau_2^{[2]}\right)\right] = \cdots \\ &= \mathbb{E}\left[\tau_{n_2}^{[2]}\left(1 - \tau_{n_2+1}^{[2]}\right)\left(1 - q\tau_{n_1}^{[1]}\right)\right] \\ &= \cdots = \mathbb{E}\left[\tau_{N_2-1}^{[2]}\left(1 - \tau_{N_2}^{[2]}\right)\right] = \mathbb{E}\tau_{N_2}^{[2]}. \end{aligned}$$

Employing the MFA, the above set of $N_1 + N_2$ equations may be solved for the $N_1 + N_2$ buffer occupancies, and consequently, the throughput of the networks at steady state for any $0 \leq q \leq 1$. However, as aforementioned, the MFA neglects all the correlations between the node occupancies.

We now present a tighter approximation, which we term the partial mean-field approximation, wherein the correlations between the occupancies of nodes involved in intersections alone are neglected.[5] The basic idea behind PMFA is to "cut" the network flow into constituent linear flows, and to use the fact that the throughput across each cut (or linear segment) in the flow is the same. To this end, we present the following lemma.

Lemma 5.1. *Consider an r-TDMA-based ad hoc network with N nodes (the channel access probability for each node is $1/(N+1)$). Let p_s denote the packet success probability across each link in the network. The throughput across a cut in the network comprising n nodes with influx and outflux rates and hopping probability α, β, and p_s, respectively, (see Fig. 1) is given by*

$$T(\alpha,\beta,n) = \begin{cases} p_s/(N+1)\times \min\{\alpha,\beta\} & n=0 \\ p_s/(N+1)\times \dfrac{Z(\alpha,\beta,n-1)}{Z(\alpha,\beta,n)} & n\geq 1, \end{cases}$$

where $Z(\alpha,\beta,0)=1$ and

$$Z(\alpha,\beta,n) = \sum_{i=1}^{n} \frac{i(2n-1-i)!}{n!(n-i)!}\frac{(1/\beta)^{i+1}-(1/\alpha)^{i+1}}{1/\beta - 1/\alpha}, n\geq 1.$$

Proof. Proving the case $n=0$ is straightforward; the rate of packet flow across the cut is the minimum of the influx and outflux rates, multiplied by the channel access and success probabilities ($1/(N+1)$ and p_s, respectively). For the case $n \geq 1$, the throughput across the flow is (7)

$$T(\alpha,\beta,n) = \frac{p_s \mathbb{E}\tau_n}{N+1} = \frac{p_s}{N+1}\frac{\langle W|C^{N-1}|V\rangle}{\langle WC^N|V\rangle}.$$

From [14, (39)], the lemma is established. □

We now show how to use the PMFA framework to evaluate the throughput for the multihop network shown in Fig. 12. First, we cut each flow across $\mathrm{S} \rightarrow \mathrm{D}$ at the common relay node R to form two line network flows. Thus, the flow $\mathrm{S}_1 \rightarrow \mathrm{D}_1$ is split into flows $\mathrm{S}_1 \rightarrow \mathrm{R}_{n_1}$ and $\mathrm{R}_{n_1} \rightarrow \mathrm{D}_1$. Now, the flow $\mathrm{S}_1 \rightarrow \mathrm{R}_{n_1}$ may be modeled as a line network flow across $n_1 - 1$ relay nodes (considering R_{n_1} as the destination node for that flow); it has an influx rate of 1 and an effective outflux rate of $\beta_{\text{eff}}^{(1)} = 1 - \mathbb{E}\tau_{n_1}^{[1]}$. Likewise, for the latter flow spanning $N_1 - n_1$ relays (through nodes R_{n_1} to D_1), the effective influx rate is $\alpha_{\text{eff}}^{(1)} = \mathbb{E}[\tau_{n_1}^{[1]}(1-(1-q)\tau_{n_2}^{[2]})]$, and the outflux rate is 1. Since the throughput across each cut is the same, we have

$$T\big(1,\beta_{\text{eff}}^{(1)},n_1\big) = T\big(\alpha_{\text{eff}}^{(1)},1,N_1-n_1\big). \tag{24}$$

Similarly, considering the second flow, we obtain

$$T\big(1,\beta_{\text{eff}}^{(2)},n_2\big) = T\big(\alpha_{\text{eff}}^{(2)},1,N_2-n_2\big), \tag{25}$$

where $\beta_{\text{eff}}^{(2)} = 1 - \mathbb{E}\tau_{n_2}^{[2]}$ and $\alpha_{\text{eff}}^{(2)} = \mathbb{E}[\tau_{n_2}^{[2]}(1-q\tau_{n_1}^{[1]})]$.

5. The PMFA gives exact performance results in networks without intersections, i.e., for a linear flow of packets. The MFA, on the other hand, is fairly inaccurate [14].

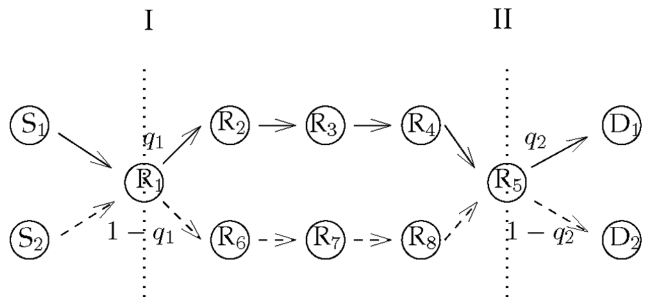

Fig. 13. A toy example consisting of two multihop flows $\mathrm{S}_1 \rightarrow \mathrm{D}_1$ and $\mathrm{S}_2 \rightarrow \mathrm{D}_2$. The packet routing priorities at the common relay nodes R_1 and R_5 are q_1 and q_2, respectively. The dotted lines I and II represent two cuts along the flow.

One may then use (24) and (25) in conjunction with Lemma 5.1 to solve for the two unknowns $\mathbb{E}\tau_{n_1}^{[1]}$ and $\mathbb{E}\tau_{n_2}^{[2]}$, and subsequently evaluate the throughputs across the two flows.

5.5 A Toy Example

We now describe how to use the PMFA to approximate the throughput performance of networks with arbitrary topologies. As intuitively expected, the PMFA method outperforms the MFA method. For the purpose of illustration, we consider a simple example comprising two six-hop flows across two common relays (see Fig. 13). The packet routing priorities for the first flow $\mathrm{S}_1 \rightarrow \mathrm{D}_1$ at the common relay nodes R_1 and R_5 are q_1 and q_2, respectively. We evaluate the throughput only for the first flow; the computation of the throughput of the second flow is quite similar.

The main idea to use is that for each flow, the throughput into a common relay node equals the throughput out of it. Accordingly, we make some "cuts" along the multihop flow, and equate the throughputs across the constituent linear flows. For the toy example shown in Fig. 13, we make two cuts I and II along the flow.

Now, for notational convenience, let $\mathbb{E}\tau_1^{[1]} = x$, $\mathbb{E}\tau_5^{[1]} = y$, $\mathbb{E}\tau_1^{[2]} = z$, and $\mathbb{E}\tau_5^{[2]} = w$. Since the rate of packet flow across each cut is the same, we obtain for the two flows, $\mathrm{S}_1 \rightarrow \mathrm{D}_1$ and $\mathrm{S}_2 \rightarrow \mathrm{D}_2$,

$$\begin{aligned} T(1,1-x,1) &= T(x(1-(1-q_1)z),1-y,3), \\ T(1,1-x,1) &= T(y(1-(1-q_2)w),1,1), \end{aligned}$$

and

$$\begin{aligned} T(1,1-z,1) &= T(z(1-q_1x),1-w,3), \\ T(1,1-z,1) &= T(w(1-q_2y),1,1), \end{aligned}$$

respectively. The above four equations may be solved to obtain the unknowns x, y, z, and w. The channel access probability for each node is $1/10$; the steady-state throughput is $T^{[1]} = p_s(1-x)/10$.

Alternatively, one may use the MFA to evaluate the throughput across the first flow at steady state. For simplicity of notation, let $\mathbb{E}\tau_1^{[1]} = x_1$, $\mathbb{E}\tau_2^{[1]} = x_2$, $\mathbb{E}\tau_3^{[1]} = x_3$, $\mathbb{E}\tau_4^{[1]} = x_4$, $\mathbb{E}\tau_5^{[1]} = x_5$, $\mathbb{E}\tau_1^{[2]} = x_6$, $\mathbb{E}\tau_2^{[2]} = x_7$, $\mathbb{E}\tau_3^{[2]} = x_8$, $\mathbb{E}\tau_4^{[2]} = x_9$, and $\mathbb{E}\tau_5^{[2]} = x_{10}$. We obtain the following 10 equations:

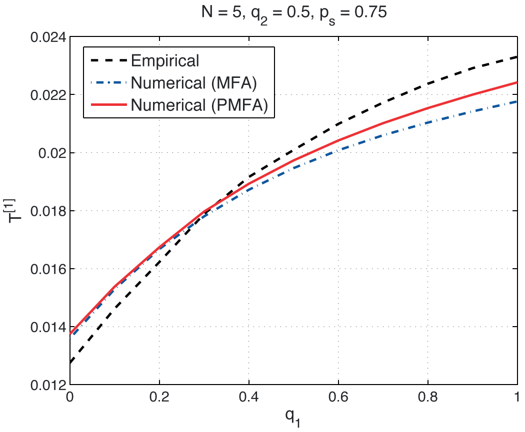

Fig. 14. Steady-state throughput across the first flow, $T^{[1]}$ versus q_1 for $q_2 = 0.5$ and $p_s = 0.75$. The results obtained numerically using the PMFA framework (solid line) closely approximate the empirical results, and is more accurate than the MFA approach (dashed line).

$$
\begin{aligned}
1 - x_1 &= x_1(1 - x_2)(1 - (1 - q_1)x_6) \\
1 - x_1 &= x_2(1 - x_3) \\
1 - x_1 &= x_3(1 - x_4) \\
1 - x_1 &= x_4(1 - x_5) \\
1 - x_1 &= x_5(1 - (1 - q_2)x_{10}) \\
1 - x_6 &= x_6(1 - x_7)(1 - q_1x_1) \\
1 - x_6 &= x_7(1 - x_8) \\
1 - x_6 &= x_8(1 - x_9) \\
1 - x_6 &= x_9(1 - x_{10}) \\
1 - x_6 &= x_{10}(1 - q_2x_5),
\end{aligned}
$$

which may be solved numerically to obtain the steady-state occupancies for any q_1, q_2. Again, we have $T^{[1]} = (1 - x_1)p_s/10$. Compared to the PMFA method, the MFA approach, however, neglects correlations between the occupancies of all pairs of nodes, and in particular, between the occupancies of nodes R_2, R_3, and R_4.

Fig. 14 plots the (steady state) throughput $T^{[1]}$ across the first flow versus q_1 for $q_2 = 0.5$, obtained upon using both the MFA and PMFA approaches, as well as simulation results. The plots show that the throughput evaluation from the PMFA framework closely matches the empirical result and is more accurate compared to the results yielded by the MFA approach, in particular, for high q_1.

6 Conclusions

In this paper, we have employed ideas from the TASEP literature to study multihop networks with random access. Specifically, we have showcased the MPA framework as a handy tool for characterizing the packet delay pmfs and the throughput performance for a linear multihop flow. We also introduced the PMFA framework, a more accurate version of the MFA, which helps quantify the throughput performances of complex ad hoc network models comprising intersecting flows.

The TASEP particle-flow model bridges the gap between statistical mechanics and wireless networking. It is useful for providing closed-form expressions for the average end-to-end delay and throughput of the multihop line network and has the advantage of obviating the cumbersome queuing-theoretic analysis. Furthermore, the results obtained are scalable with the number of nodes and thus can provide helpful insights into the design of wireless networks. We wish to promote TASEPs as a useful tool to analyze the performance of ad hoc networks and hope that this introductory work instigates interest in solving other relevant wireless networking problems employing ideas from statistical mechanics.

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Appendix

Proof of Lemma 4.3

Proof: Using the fact that $C = D + E$, we obtain

$$\begin{aligned}\Delta_{i,j}^{(N)} &= \frac{\langle W|C^{i-1}D^{j-2}(C-E)C^{N-i-j}|V\rangle}{\langle W|C^{N-1}|V\rangle} \\ &= \frac{\langle W|C^{i-1}D^{j-2}C^{N-i-j+1}|V\rangle}{\langle W|C^{N-1}|V\rangle} \\ &- \frac{\langle W|C^{i-1}D^{j-3}DEC^{N-i-j}|V\rangle}{\langle W|C^{N-1}|V\rangle} \\ &\overset{(a)}{=} \Delta_{i,j-1}^{(N)} - \frac{\langle W|C^{i}D^{j-3}C^{N-i-j+1}|V\rangle}{p_s\langle W|C^{N-1}|V\rangle} \\ &= \Delta_{i,j-1}^{(N)} - \Delta_{i,j-2}^{(N-1)}\frac{\langle W|C^{N-2}|V\rangle}{p_s\langle W|C^{N-1}|V\rangle},\end{aligned}$$

which is equivalent to (13), upon using property (3b). Here, (a) is derived using (10) and (3a). $\square$

Proof of Lemma 4.4

Proof: The proof involves induction. Using (11) and (12) in (13), we obtain for the case $j = 2$,

$$\Delta_{i,2}^{(N)} = \mathbb{E}\tau_i^{(N-2)}\frac{\eta(N-2)}{\eta(N-1)},$$

which satisfies (14). Similarly, using (13) for $j = 3$, we have

$$\Delta_{i,3}^{(N)} = \mathbb{E}\tau_i^{(N-2)}\frac{\eta(N-2)}{\eta(N-1)} - \frac{\eta(N-3)}{\eta(N-1)},$$

which conforms with (14).

Suppose that (14) is valid for the cases $j = m, m-1$, $m > 2$. Now, consider the case $j = m+1$. From (13), we have

$$\begin{aligned}\Delta_{i,m+1}^{(N)} &= \Delta_{i,m}^{(N)} - \Delta_{i,m-1}^{(N-1)}\eta(N-2)/\eta(N-1) \\ &= \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\frac{\eta(N-k-2)}{\eta(N-1)}\times \\ &\quad\left[\binom{m-k-2}{k}\mathbb{E}\tau_i^{(N-k-2)} + \binom{m-k-2}{k-1}\right] \\ &\quad -\frac{\eta(N-2)}{\eta(N-1)}\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^k\frac{\eta(N-k-3)}{\eta(N-2)}\times \\ &\quad\left[\binom{m-k-3}{k}\mathbb{E}\tau_i^{(N-k-3)} + \binom{m-k-3}{k-1}\right] \\ &= \binom{m-2}{0}\frac{\eta(N-2)}{\eta(N-1)}\mathbb{E}\tau_i^{(N-2)} + \sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k \\ &\quad\frac{\eta(N-k-2)}{\eta(N-1)}\left[\binom{m-k-2}{k}\mathbb{E}\tau_i^{(N-k-2)} + \right. \\ &\quad\left.\binom{m-k-2}{k-1}\right] + \sum_{k=1}^{\lfloor\frac{m-2}{2}\rfloor+1}(-1)^k\frac{\eta(N-k-2)}{\eta(N-1)} \\ &\quad\left[\binom{m-k-2}{k-1}\mathbb{E}\tau_i^{(N-k-2)} + \binom{m-k-2}{k-2}\right] \\ &= \binom{m-2}{0}\frac{\eta(N-2)}{\eta(N-1)}\mathbb{E}\tau_i^{(N-2)} + \sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k \\ &\quad\frac{\eta(N-k-2)}{\eta(N-1)}\left[\left[\binom{m-k-2}{k} + \binom{m-k-2}{k-1}\right]\right. \\ &\quad\left.\mathbb{E}\tau_i^{(N-k-2)} + \left[\binom{m-k-2}{k-1} + \binom{m-k-2}{k-2}\right]\right] \\ &\quad + (-1)^{m/2}\frac{\eta(N-m/2-2)}{\eta(N-1)}1[m\text{ is even}],\end{aligned}$$

where the last term involves an indicator function since it occurs only when $\lfloor(m-2)/2\rfloor + 1 \neq \lfloor(m-1)/2\rfloor$, i.e., when m is even. Using the identity

$$\binom{r}{s} + \binom{r}{s-1} = \binom{r+1}{s}, \tag{21}$$

we obtain

$$\begin{aligned}\Delta_{i,m+1}^{(N)} &= \sum_{k=0}^{\lfloor m/2\rfloor}(-1)^k\frac{\eta(N-k-2)}{\eta(N-1)}\times \\ &\quad\left[\binom{m-k-1}{k}\mathbb{E}\tau_i^{(N-k-2)} + \binom{m-k-1}{k-1}\right].\end{aligned} \tag{22}$$

We see that (14) is valid for the case $j = m+1$ as well. By induction, the lemma holds. $\square$

Proof of Theorem 4.7

Proof: We condition on the event e_{i,j_1,j_2}, which happens w.p. $\kappa_{i,j_1,j_2}^{(N)}$. For clarity, we treat the following two cases separately.

Case 1: $0 \leq j_1 \leq N - i - 1$
Since $D_i = k$ by assumption, at least j_1 packet hops (for the ones at nodes $i + j_1, \dots i + 1$, in that order) occur within $k - 1$ slots. In addition, $D_{i+1} = \ell$, thus the packets at nodes $i + 1, \dots i + j_1$ hop out twice, which also means that j_2 other packet hops (from nodes $i + j_1 + 2, \dots, i + j_1 + j_2 + 1$ to their respective adjacent nodes) occur, all within $k + \ell - 2$ slots. The flow length of N nodes also places a constraint on the possible values that j_1 and j_2 can take. Thus, the following hold:

(i) $j_1 \leq k - 1$.
(ii) $2j_1 + j_2 \leq k + \ell - 2$.
(iii) $i + j_1 + j_2 + 1 \leq N$.

Equivalently, we have

- $0 \leq j_1 \leq \min\{k - 1, N - i - 1\}$, and
- $0 \leq j_2 \leq \min\{k + \ell - 2j_1 - 2, N - i - 1 - j_1\}$.

The conditional joint pmf $\mathbb{P}(D_{i+1} = \ell, D_i = k \mid e_{i,j_i,j_2})$ is equal to the sum of the probabilities of having j successful packet hops, $j_1 \leq j < 2j_i + j_2$, occurring in $k - 1$ time slots, then the packet at node i hopping successfully to node $i + 1$ in the k^{th} time slot, then $2j_1 + j_2 - j$ successful transmissions occurring in $\ell - 1$ slots, and finally, the packet at node $i + 1$ hopping to node $i + 2$ in the $k + \ell$ th time slot.

Case 2: $j_1 = N - i$
For this case, when a packet arrives at node i, the configuration of the nodes $i+1, \dots, N$ is simply $(\tau_{i+1}, \dots \tau_N) = (1, \dots, 1)$. As in Case 1, at least j_1 packet hops (for the ones at nodes $i + j_1, \dots i + 1$, in that order) occur within $k - 1$ slots. However, once the packet at node N is delivered to the destination, it does not hop further. Thus, $D_{i+1} = \ell$ would mean that $2j_1 - 1$ (and not $2j_1$) successful transmissions must occur in $k + \ell - 2$ time slots. The following constraints hold:

(i) $j_1 \leq k - 1$.
(ii) $2j_1 - 1 \leq k + \ell - 2$.

Putting together (i) and (ii), $0 \leq j_1 \leq \min\{k - 1, (k + \ell - 1)/2\}$.

The conditional joint pmf $\mathbb{P}(D_{i+1} = \ell, D_i = k \mid e_{i,j_i,j_2})$ is obtained by adding up the probabilities of having j successful packet hops, $j_1 \leq j < 2j_i - 1$, occurring in $k - 1$ time slots, then the packet at node i hopping successfully to node $i + 1$ in the k^{th} time slot, then $2j_1 - 1 - j$ successful transmissions occurring in $\ell - 1$ slots, and lastly, the packet at node $i + 1$ hopping to node $i + 2$ in the $k + \ell$ th time slot.

Summing up $\mathbb{P}(D_{i+1} = \ell, D_i = k, e_{i,j_i,j_2})$ over all possible values of j_1 and j_2 considering both the cases yields the joint pmf, which is the same as (18). $\square$