Modeling Interference in Finite Uniformly Random Networks

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Introduction

- A sensor network is (often) formed by scattering nodes randomly over an area that needs to be monitored.
- Distribution of nodes is ubiquitously modeled as a Poisson point process (PPP).
  - Ease of analysis.
  - Provides useful insights.
- This assumption is not practical in many cases.
- **Our goal** : Study the more realistic "binomial network", and characterize the interference in such a system.

The Poisson Network

For a PPP of intensity λ,

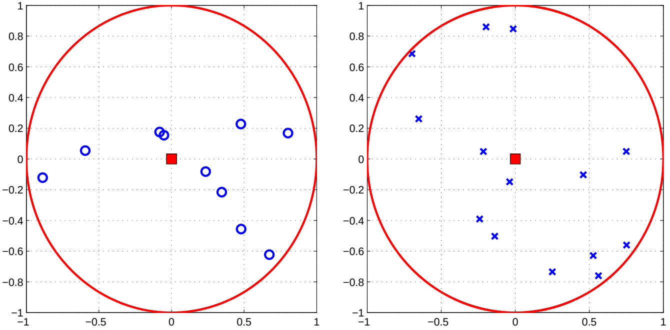
- The number of nodes in any given set of Lebesgue measure V is Poisson distributed with mean λV.

$$\Pr(\Phi(V) = k) = e^{-\lambda V} \frac{(\lambda V)^k}{k!}.$$

- The number of nodes in disjoint sets are independent of each other.

Impracticalities of the PPP Assumption

- Networks are formed by usually scattering a *fixed* (and *finite*) number of nodes in a (finite) given area.
- The point process formed is non-stationary and usually non-isotropic.
- The number of nodes in disjoint sets are not independent but governed by a multinomial distribution.



(Left) A realization of 10 sensor nodes uniformly randomly distributed in a circular area of unit radius. (Right) The Poisson network with the same density (λ =3.18) has 14 nodes. The shaded box at the origin represents the base station.

The Binomial Point Process (BPP)

- A BPP $\Phi_W^{(N)}$ is formed as a result of distributing N points independently and uniformly in a compact set $W \subset \mathbb{R}^d$.
- For a Borel subset A of W, the number of points falling in A is binomial(n, p) with parameters $n = N$ and $p = \nu_d(A)/\nu_d(W)$, where $\nu_d(\cdot)$ is the standard d-dimensional Lebesgue measure.
- Conditioned on the total number of nodes in a given volume, the PPP transforms into the BPP.
- Binomial networks are those whose nodes are distributed as a BPP.

System Model

- N transmitting nodes uniformly randomly distributed in a d-dimensional ball of radius R.
- Density of the process $\lambda = N/(c_d R^d)$, where $c_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$.
- Each node collects data and transmits it towards a base station positioned at origin.
- MAC scheme : slotted ALOHA.
- Our paper studies the interference power at the center of the network.

Channel Model

- Attenuation in the channel : modeled as a product of
  - Large-scale path loss decay (exponent γ).
  - Amplitude fading H is m-Nakagami distributed (general case).
- $m = 1$: Rayleigh fading; $m = \infty$: constant gain.
- Power fading variable : $G = H^2$.
- Moments : Assume parameter Ω

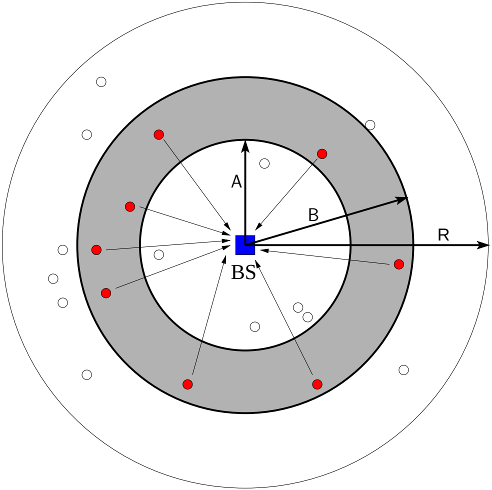
$$\mathbb{E}_G[G^n] = \frac{\Gamma(m+n)}{m^n \Gamma(m)} \Omega^n.$$

- Mean $= \Omega$ and variance $= \Omega^2/m$.
- All the results obtained are for an "average network".

Interference Modeling

- Fact : the pdf of the interference can be expressed in closed form for a very few number of cases.
  - Lévy distribution.
  - Gaussian distribution.
- Work around this issue by dealing with the moment generating function (MGF).
- **Problem** : characterize the interference at the origin resulting only from the nodes in the annular region with inner radius A and outer radius B $(0 \leqslant A < \mathrm{B} \leqslant \mathrm{R})$.
- Letting $\mathrm{N} \to \infty$ and $\mathrm{R} \to \infty$ while keeping λ constant, the BPP transforms to the PPP.

Problem Depiction



The system model for the 2-dimensional case. The red-colored nodes inside the shaded region are the transmitters considered.

Interference Modeling (Contd.)

- The MGF of the interference at the origin is given by

$$M_I(s) = \left(1 - \frac{\lambda}{N}\mathbb{E}_G \underbrace{\left[\int_A^B \left(1 - \exp\left(-sGr^{-\gamma}\right)\right) \mathrm{d}c_d r^{d-1}\mathrm{d}r\right]}_{D(s)}\right)^N.$$

- $D(s)$ can be simplified as

$$\begin{aligned} D(s) &= c_d B^d \left[1 - e^{-sGB^{-\gamma}}\right] - c_d A^d \left[1 - e^{-sGA^{-\gamma}}\right] + c_d\,(sG)^{d/\gamma} \\ &\quad \Gamma\left(1 - \frac{d}{\gamma}, sGB^{-\gamma}\right) - c_d\,(sG)^{d/\gamma}\,\Gamma\left(1 - \frac{d}{\gamma}, sGA^{-\gamma}\right), \end{aligned}$$

where $\Gamma(a, z)$ is the upper incomplete Gamma function, defined as

$$\Gamma(a, z) = \int_z^\infty \exp(-t)t^{a-1}\mathrm{d}t.$$

Cumulants of the Interference

- The n^{th} cumulant of the interference is defined as

$$C_n = (-1)^n \frac{d^n}{ds^n} \ln M_I(s)\Big|_{s=0}.$$

- C_1 is the mean and C_2 the variance.
- The cumulants can be expressed recursively as

$$C_n = NT_n - \sum_{i=1}^{n-1} \binom{n-1}{i-1} C_i T_{n-i},$$

where

$$T_n := \begin{cases} \frac{d}{R^d} \mathbb{E}_G [G^n] \left[\frac{B^{d-n\gamma} - A^{d-n\gamma}}{d-n\gamma} \right] & , \gamma \neq \frac{d}{n} \\ \frac{d}{R^d} \mathbb{E}_G [G^n] \ln \left(\frac{B}{A} \right) & , \gamma = \frac{d}{n}. \end{cases}$$

Cumulants of the Interference (Contd.)

- Cumulants for a Poisson network :

$$C_n = \lambda d c_d \mathbb{E}_G[G^n] \frac{B^{d-n\gamma} - A^{d-n\gamma}}{d - n\gamma}, \quad \gamma \neq \frac{d}{n}.$$

- The ratio of the n^{th} cumulants with and without fading is

$$\frac{C_n|_{m=m}}{C_n|_{m=\infty}} = \frac{(m+n-1)!}{m^n (m-1)!}.$$

- The mean interference is independent of m.
- The variance of the interference doubles for Rayleigh fading compared to the case of no fading. †

† J. Venkataraman and M. Haenggi, "Optimizing the Throughput in Random Wireless Ad Hoc Networks," *42st Annual Allerton Conference on Communication, Control, and Computing, (Monticello, IL)*, Oct. 2004.

Special Cases : α-stable distribution

- Poisson network of density λ ($A = 0, B \to \infty$, $0 < d < \gamma$)
  - Interference approaches stable distribution, with stability index $\alpha = d/\gamma$. †
  - The MGF is of the form

$$M_I(s) = \exp\left(-\lambda c_d \mathbb{E}_G\left[G^{d/\gamma}\right]\Gamma\left(1 - d/\gamma\right) s^{d/\gamma}\right).$$

  - The interference never converges to a Gaussian distribution.
  - All the moments of the interference are infinite.
  - $\alpha = 0.5$: "Lévy" distribution. †

$$P_I(x) = \sqrt{\frac{\beta}{\pi}} x^{-3/2} \exp(-\beta/x), \quad x \geqslant 0,$$

where $\beta = (\pi\lambda^2 c_d^2 \mathbb{E}_G^2[G^{1/2}])/4$.

† G. Samorodnitsky and M. S. Taqqu, "Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance," Chapman and Hall, 1994.

Special Cases : Gaussian distribution

Asymptotic results ($N \to \infty$)

- $A > 0, B < \infty$
  - All the moments are finite.
  - CLT is valid and interference approaches a Gaussian r.v.

$$P_I(x) \to \mathcal{N}(C_1, C_2) = \frac{1}{\sqrt{2\pi C_2}} \exp(-(x - C_1)^2/2C_2).$$

- $A = 0, B < \infty$
  - The interference approaches a Gaussian for $d > 2\gamma$.
  - For $1/2 \leqslant \gamma/d < 1$, the mean interference is finite while its variance is unbounded.
- $A > 0, B \to \infty, d < \gamma$

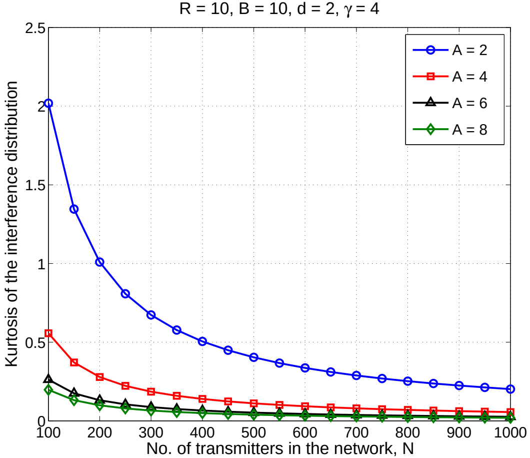
Convergence to a Gaussian

- How fast does the interference converge to a Gaussian?
- Kurtosis (excess) is a good parameter to characterize this.
- It is a measure of the peakedness of the pdf of a real-valued random variable.
- It is defined as

$$\kappa(I) = \frac{\mathbb{E}\left[(I - \mu_I)^4\right]}{\sigma_I^4} - 3 = \frac{C_4}{C_2^2}.$$

- Kurtosis for a Gaussian distribution $= 0$.

Convergence to a Gaussian (contd.)



Kurtosis of the interference distribution for different values of A.

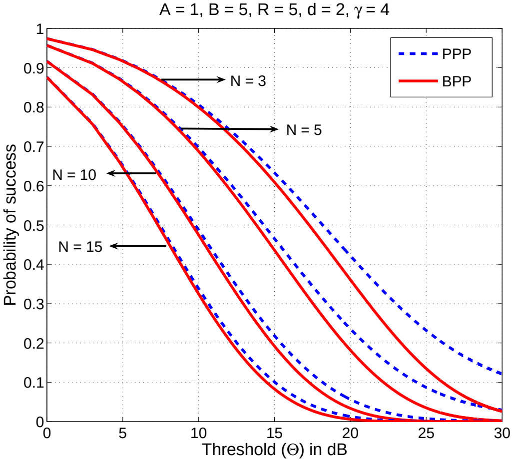
Network Outage

- Assume that background noise is much smaller than the interference.
- An outage $\mathcal{O}$ is defined to occur if the SIR is lower than a predetermined threshold Θ.
- Assume that the desired transmitter node is located at unit distance from the origin and is transmitting at unit power.
- Under Rayleigh fading, the received power is exponential with unit mean.
- The outage probability $\Pr(\mathcal{O})$ is

$$\begin{aligned}\Pr(\mathcal{O}) &= \mathbb{E}_I\left[\Pr(G/I < \Theta \mid I)\right] = \mathbb{E}_I\left[1 - \exp(-I\Theta)\right] \\ &= 1 - M_I(\Theta).\end{aligned}$$

- The probability of success p_s is equal to $M_I(\Theta)$.

Network Outage (contd.)



Comparison of success probabilities for Poisson and binomial networks for different values of N under Rayleigh fading.

Summary and Conclusion

- Characterized the interference in a binomial network.
- Derived a closed-form expression for the MGF and used it to calculate the interference moments.
- Studied the asymptotic convergence to a Gaussian distribution.
- Using the Poisson model in analyses provides an overly optimistic estimate of the network's outage performance, especially when the number of interferers is small.