Modeling Interference in Uniformly Random Wireless Networks: Theory and Applications

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Outline

Part 1: Theory

- Characterize the interference in a finite uniformly random network.
    - Specifically, derive the moment generating function (MGF).
    - Compute the interference moments.
- Study asymptotic behavior of the network interference.
- Extend results to an infinite Poisson network.

Part 2: Applications

- Evaluation of the outage performance.
- Path loss exponent (PLE) estimation.

Interference in Wireless Networks

- An wireless ad hoc or sensor network is usually formed by randomly deployed nodes with self-organizing capabilities.
- Owing to concurrent transmissions, interference is commonly experienced in such systems.
- Modeling interference is often mathematically intractable and may not offer much insight.
- Inclusion of interference in analyses is often circumvented by
  - Taking noise power to be much larger than the interference power.
  - Employing TDMA/FDMA/CDMA/SDMA.
  - Assuming that the interference is much lower than the signal strength.
  - Using successive interference cancellation.

Uniformly Random Networks and the PPP

- Prior work related to modeling interference has commonly considered the nodal distribution to be the Poisson point process (PPP).
  - Analysis is tractable.
  - Useful insights are obtainable.
- For a homogeneous PPP of intensity λ,
  - The number of nodes in any given set of Lebesgue measure V is Poisson distributed with mean λV.

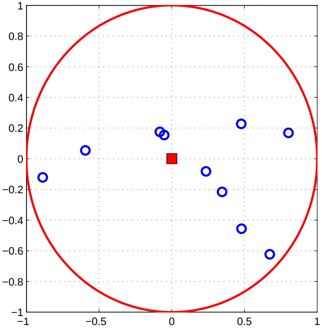
$$\Pr(\Phi(V) = k) = e^{-\lambda V} \frac{(\lambda V)^k}{k!}.$$

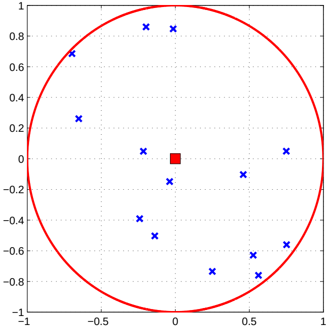
  - The number of nodes in disjoint sets are independent of each other.

Impracticalities of the PPP Assumption

- The PPP assumption is not practical in many cases.
  - Networks are formed by usually scattering a *fixed* (and *finite*) number of nodes in a (finite) given area.
  - The point process formed is usually non-stationary and non-isotropic.
  - The number of nodes in disjoint sets are not independent but governed by a multinomial distribution.
- We study the more realistic binomial network, and characterize the interference in such a system.

Impracticalities of the Poisson Assumption (contd.)





(Left) A realization of 10 sensor nodes uniformly randomly distributed in a circular area of unit radius. (Right) This particular realization of the Poisson network with the same density (λ =3.18) has 14 nodes.

The Binomial Point Process (BPP)

- A binomial point process is formed as a result of distributing N points independently and uniformly in a compact set $W \subset \mathbb{R}^d$.
- For a Borel subset V of W,

$$\Pr(\Phi(V) = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

  where $n = N$ and $p = \nu_d(V)/\nu_d(W)$, and $\nu_d(\cdot)$ is the standard d-dimensional Lebesgue measure.
- Binomial networks are those whose nodes are distributed as a BPP.

Interference Characterization

System Model

- N nodes uniformly randomly distributed in a d-dimensional ball of radius R.
- Density of the process $\lambda = N/(c_d R^d)$, where $c_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$.
- MAC scheme: slotted ALOHA with contention probability p.
- We characterize the interference power measured at the origin.

Channel Model

- Attenuation in the channel: modeled as a product of
  - Large-scale path loss decay (exponent γ).
  - Amplitude fading H which is assumed to be m-Nakagami distributed ($m \geqslant 0.5$).
- $m = 1$: Rayleigh fading; $m = \infty$: constant gain.
- Power fading variable: $G = H^2$.
- Moments:

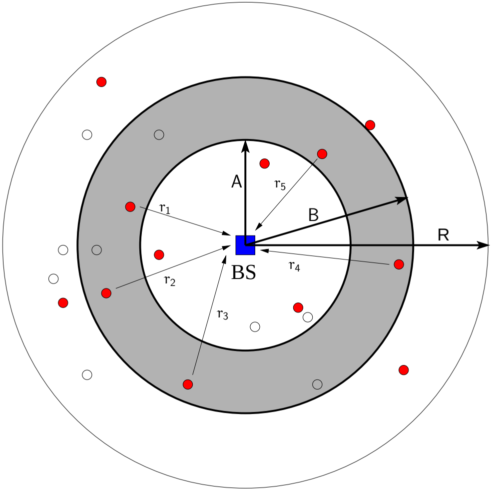
$$\mathbb{E}_G[G^n] = \frac{\Gamma(m+n)}{m^n \Gamma(m)}.$$

- Mean $= 1$ and variance $= 1/m$.
- $P_y = g r^{-\gamma} P_x$.
- All the results obtained are for an "average network".

Interference Modeling

- Fact: the pdf of the interference can be expressed in closed form for a small number of cases. The resulting distributions are
  - The Lévy distribution.
  - The Gaussian distribution.
- Work around this issue by dealing with the MGF.
- Problem: characterize the interference at the origin resulting only from the nodes in the annular region with inner radius A and outer radius B $(0 \leqslant A < B \leqslant R)$.
- Letting $N \to \infty$ and $R \to \infty$ while keeping λ constant, the BPP transforms to the PPP.

Problem Depiction



The system model for the 2-dimensional case. The transmitters are red.

We characterize $I = \sum_i \frac{g_i}{r_i^\gamma}$.

Interference Modeling (contd.)

- The MGF of the interference at the origin is given by

$$M_I(s) = \left(1 - \frac{\lambda p}{N}\mathbb{E}_G\underbrace{\left[\int_A^B \left(1 - \exp\left(-sGr^{-\gamma}\right)\right) dc_d r^{d-1} dr\right]}_{D(s)}\right)^N.$$

- $D(s)$ can be expressed as

$$D(s) = c_d B^d \left[1 - e^{-sGB^{-\gamma}}\right] - c_d A^d \left[1 - e^{-sGA^{-\gamma}}\right] + c_d (sG)^{d/\gamma} \Gamma\left(1 - \frac{d}{\gamma}, sGB^{-\gamma}\right) - c_d (sG)^{d/\gamma} \Gamma\left(1 - \frac{d}{\gamma}, sGA^{-\gamma}\right),$$

where $\Gamma(a, z)$ is the upper incomplete Gamma function, defined as

$$\Gamma(a, z) = \int_z^\infty \exp(-t) t^{a-1} dt.$$

Cumulants of the Interference

- The n^{th} cumulant of the interference is defined as

$$C_n = (-1)^n \frac{d^n}{ds^n} \ln M_I(s)\Big|_{s=0}.$$

- C_1 is the mean and C_2 the variance.
- Result: The cumulants can be expressed recursively as

$$C_n = NT_n - \sum_{i=1}^{n-1} \binom{n-1}{i-1} C_i T_{n-i},$$

where

$$T_n := \begin{cases} \frac{dp}{R^d}\mathbb{E}_G\left[G^n\right]\left[\frac{B^{d-n\gamma}-A^{d-n\gamma}}{d-n\gamma}\right] & ,\gamma \neq \frac{d}{n} \\ \frac{dp}{R^d}\mathbb{E}_G\left[G^n\right]\ln\left(\frac{B}{A}\right) & ,\gamma = \frac{d}{n}. \end{cases}$$

Cumulants of the Interference (contd.)

- Cumulants for a homogeneous Poisson network:

$$C_n = \lambda p d c_d \mathbb{E}_G[G^n] \frac{B^{d-n\gamma} - A^{d-n\gamma}}{d - n\gamma}, \quad \gamma \neq \frac{d}{n}.$$

- The ratio of the n^{th} cumulants with and without fading is

$$\frac{C_n}{C_n|_{m=\infty}} = \frac{(m+n-1)!}{m^n (m-1)!}.$$

- The mean interference is independent of m.
- The variance of the interference doubles for Rayleigh fading compared to the case of no fading. †

† J. Venkataraman and M. Haenggi, "Optimizing the Throughput in Random Wireless Ad Hoc Networks," *42st Annual Allerton Conference on Communication, Control, and Computing, (Monticello, IL)*, Oct. 2004.

Special Cases: The Gaussian Distribution

Asymptotic results for dense networks ($N \to \infty$ and fixed A, B)

- $A > 0, B < \infty$
  - CLT is valid and interference approaches a Gaussian r.v.

$$P_I(x) \to \mathcal{N}(C_1, C_2) = \frac{1}{\sqrt{2\pi C_2}} \exp(-(x - C_1)^2/2C_2).$$

- $A = 0, B < \infty$
  - The interference approaches a Gaussian only for $d > 2\gamma$.
- $A > 0, B = \infty$
  - The interference approaches a Gaussian only for $d < \gamma$.

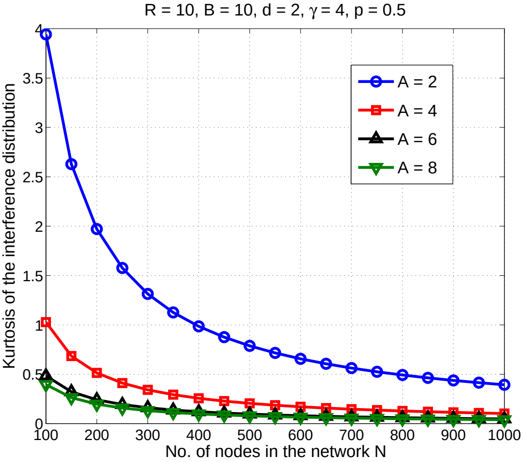
Convergence to a Gaussian

- How fast does the interference converge to a Gaussian?
- Kurtosis (excess) is a good parameter to characterize this.
- It is a measure of the peakedness of the pdf of a real-valued random variable.
- It is defined as

$$\kappa(I) = \frac{\mathbb{E}\left[(I - \mu_I)^4\right]}{\sigma_I^4} - 3 = \frac{C_4}{C_2^2}.$$

- Kurtosis for a Gaussian distribution is normalized to 0.

Convergence to a Gaussian (contd.)



Kurtosis of the interference distribution for different values of A.

Special Cases: The α-stable Distribution

- Poisson network of density λ ($A = 0, B = \infty$, $0 < d < \gamma$)
  - Interference takes the form of an stable distribution, with stability index $\alpha = d/\gamma$. †
  - The MGF is of the form

$$M_I(s) = \exp\left(-\lambda p c_d \mathbb{E}_G\left[G^{d/\gamma}\right]\Gamma\left(1 - d/\gamma\right) s^{d/\gamma}\right).$$

  - The interference never converges to a Gaussian distribution.
  - All the moments of the interference are infinite.
  - $\alpha = 0.5$: "Lévy" distribution. †

$$P_I(x) = \sqrt{\frac{\beta}{\pi}} x^{-3/2} \exp(-\beta/x), \quad x \geqslant 0,$$

where $\beta = (\pi\lambda^2 p^2 c_d^2 \mathbb{E}_G^2[G^{1/2}])/4$.

†G. Samorodnitsky and M. S. Taqqu, "Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance," Chapman and Hall, 1994.

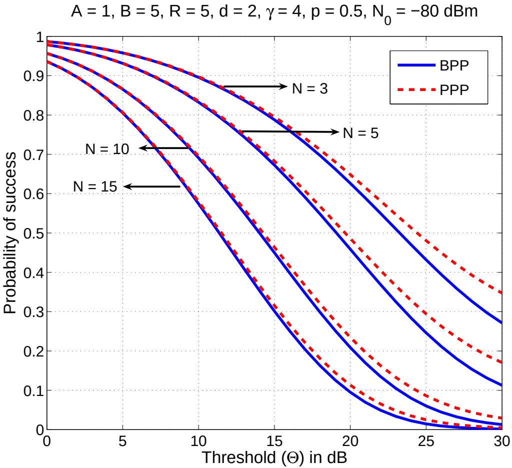
Network Outage

Network Outage

- An outage is defined to occur if the SINR at the receiver is lower than a predetermined threshold Θ.
- We determine the outage probability for a receiver placed at the origin.
- Let the transceiver pair distance be unity and the transmit power be unity.
- Assume that noise and interference are independent.
- Under Rayleigh fading, the success probability is (N_0: noise power)

$$p_s = \exp(-N_0\Theta)M_I(\Theta).$$

Network Outage (contd.)



Comparison of success probabilities for Poisson and binomial networks for different values of N under Rayleigh fading.

Path Loss Exponent Estimation

Path Loss Exponent Estimation

- Attenuation in the strength of the propagated signal: product of
  - Path loss: large-scale, deterministic.
  - Fading: small-scale, modeled as randomly varying.
- A critical issue is to characterize the path loss exponent (PLE).
  - Localization.
  - Energy-efficient transmission.
  - Handoff in cellular networks.
- This problem is not trivially solvable due to
  - Fading and interference.
  - Distances between nodes are also subject to uncertainty.

Path Loss Exponent Estimation (contd.)

- We present three different methods to estimate the PLE for large random wireless networks, based on
  - Interference moments.
  - Outage probabilities.
  - Connectivity properties.

Assumptions:

- Infinite planar network where nodes are distributed as a homogeneous PPP of density λ.
- Transmitters use slotted ALOHA channel access mechanism with contention parameter p.
- Nakagami-m fading.

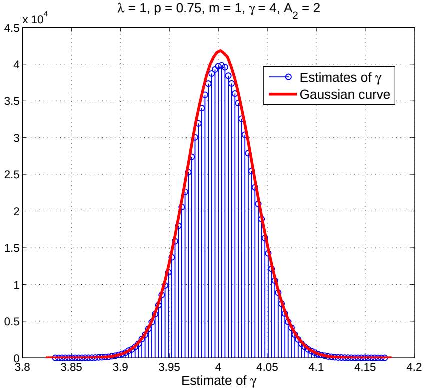
Path Loss Exponent Estimation (contd.)

- Estimation of the PLE γ is basically performed by matching the empirical and theoretic values.
- Caveat: in practice, we have access to only a single nodal realization whereas in theory, results are for the "average" network.
- Solution: ergodic property of the homogeneous PPP.
  - Statistical averages of measurable functions may be replaced by spatial averages.

Estimation Based on the Interference Moments

- Nodes need to have guard zones to ensure finite interference moments.
- With a guard zone radius of A, $\mu_I = 2\pi\lambda p \frac{A^{2-\gamma}}{\gamma-2}$.
- For a guard zone radius of $A_1 = 1$, record the values of the interference powers at several nodes $1, \ldots, N$.
- Evaluate the empirical mean interference $\mu_I' = 1/N \sum_{i=1}^{N} I_i$, and estimate γ by equating this to the theoretical value.
- Alternatively, use another guard zone radius, say A_2 and evaluate μ_I''.
- $\hat{\gamma} = 2 - \frac{\ln(\mu_I''/\mu_I')}{\ln(A_2)}$.
- This "differential" method does not need estimates of p or λ.

Estimation Based on the Interference Moments (contd.)



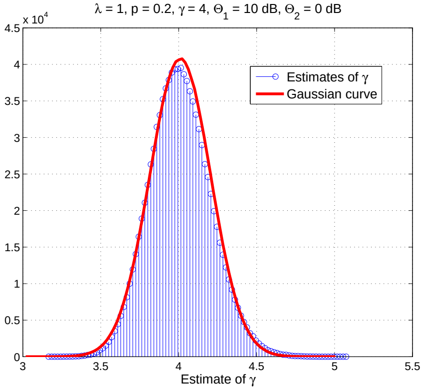
Histogram of $\hat{\gamma}$ for the estimation algorithm based on the mean interference. The error is approximately Gaussian.

Estimation Based on Outage Probabilities

- Imposing a guard zone may not be feasible, particularly when relative node location information is not available.
- When the channel is Rayleigh-faded, the PLE can also be estimated using outage probabilities.
- Recall: when $N_0 \ll I$, the success probability p_s is proportional to $\exp(-\Theta^{2/\gamma})$.
- Using a differential method,

$$\hat{\gamma} = \frac{2\ln(\Theta_1/\Theta_2)}{\ln(\ln(p_s^1)/\ln(p_s^2))}.$$

Estimation Based on Outage Probabilities (contd.)



Histogram of $\hat{\gamma}$ for the method based on outage probabilities.

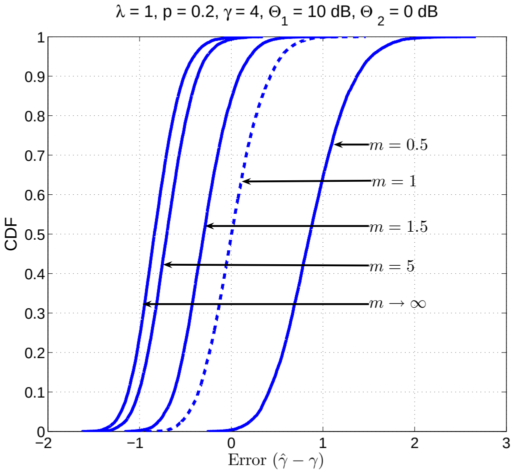
Estimation Based on Outage Probabilities (contd.)

- How is the error behavior when the fading is not actually Rayleigh ($m \neq 1$)?
- For $m \in \mathbb{N}$, the probability of success can be simplified to

$$p_s = \sum_{k=0}^{m-1} \frac{\exp(-c_3) c_3^k}{k!} \left(\frac{2}{\gamma}\right)^k,$$

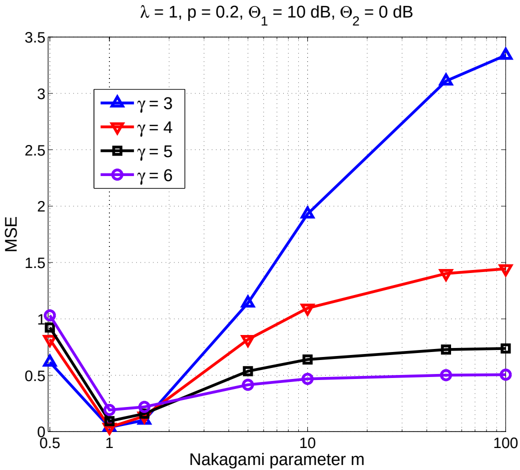
where $c_3 = \lambda p \pi \mathbb{E}_G[G^{2/\gamma}]\Gamma(1 - 2/\gamma)(\Theta m)^{2/\gamma}$.

Estimation Based on Outage Probabilities (contd.)



CDF of the error $\hat{\gamma} - \gamma$.

Estimation Based on Outage Probabilities (contd.)



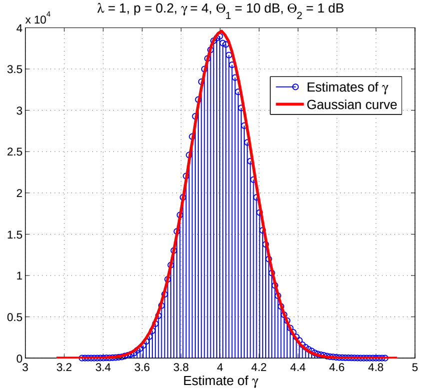
The value of the MSE versus m for different PLEs.

Estimation based on the Size of the Transmitting Set

- For any node, define its transmitting set as the as the set of nodes it receives a packet from, in a given timeslot.
- That is, node x is in the transmitting set of node y if the SINR at y due to x's signal is $\geqslant \Theta$.
- Result: Under the conditions of Rayleigh fading and $N_0 \ll I$, the size of the transmitting set is Poisson distributed with parameter $\bar{N}_T = (\Gamma(1 + 2/\gamma)\Gamma(1 - 2/\gamma)\Theta^{2/\gamma})^{-1}$.
- This algorithm is based on matching the theoretic and practical values of the mean cardinality of the transmitting set.
- An unbiased estimate is

$$\hat{\gamma} = \frac{2 \ln(\Theta_2/\Theta_1)}{\ln(\bar{N}_T^1/\bar{N}_T^2)}.$$

Estimation based on the Size of the Tx Set (contd.)



Histogram of $\hat{\gamma}$ for the estimation algorithm based on the cardinality of the transmitting set.

Estimation based on the Size of the Tx Set (contd.)

- How critical is the Rayleigh fading assumption?
- For $m \in \mathbb{N}$, the size of the Tx set is Poisson with mean

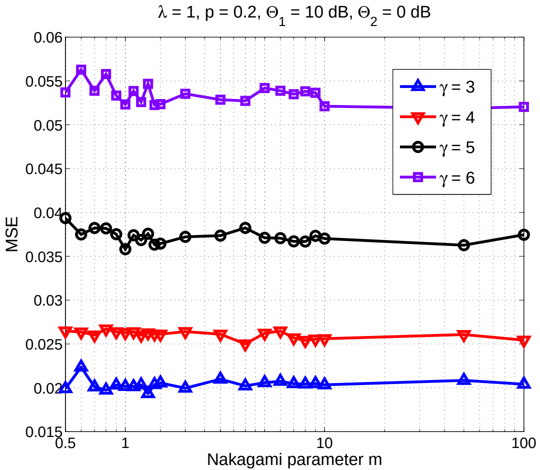
$$\bar{N}_T = \frac{\Gamma(m)\left(1 - \left(\frac{2}{\gamma}\right)^m\right)}{\Gamma(m + \frac{2}{\gamma})\Gamma(2 - \frac{2}{\gamma})\Theta^{2/\gamma}}.$$

- We surmise that $\bar{N}_T$ is inversely proportional to $\Theta^{2/\gamma}$ for all $m \in \mathbb{R}$.
- Remarkably, using the differential method, we still obtain

$$\frac{\bar{N}_T^1}{\bar{N}_T^2} = \left(\frac{\Theta_2}{\Theta_1}\right)^{2/\gamma}.$$

- Performance is insensitive to m.

Estimation based on the Size of the Tx Set (contd.)



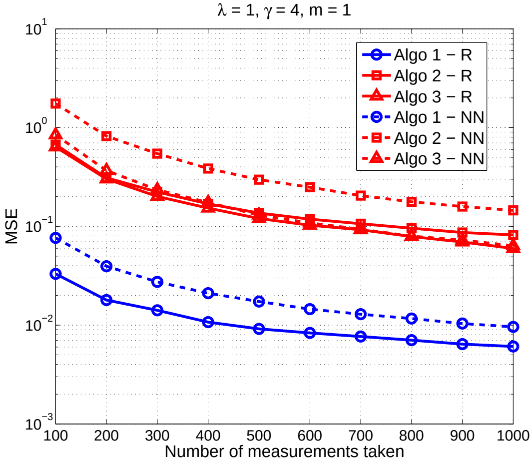
The value of the MSE versus m for different PLEs.

Comparison of the Algorithms

Success of the algorithms is critically determined by

- The number of survey points.
  - Small N: low accuracy.
  - large N: expensive survey process.
- The order in which nodes are chosen.
  - Locally as subsequent nearest neighbors (NN): correlated measurements, slow convergence.
  - Randomly (R): fast convergence, but large overhead.

Comparison of the Algorithms (contd.)



Comparison of the MSE performance of the three algorithms.

Summary

- Analytically characterized the interference in uniformly random networks.
- Derived a closed-form expression for the MGF and used it to calculate the interference moments.
- Studied the asymptotic convergence to Lévy, Gaussian distributions.
- Evaluated the outage performance of the random wireless network.
- Described three different algorithms for PLE estimation and provided simulation results on the estimation errors.

Concluding Remarks

- It is important to distinguish between the Poisson network and the binomial network. Using the Poisson model in analyses provides an overly optimistic estimate of the network's outage performance, especially when the number of interferers is small and Θ is high.
- The PLE estimation problem is both practically relevant and mathematically challenging.
- The PLE value changes depending on the terrain category and the environmental conditions, and hence cannot be assumed to be a constant over the entire network. However, our algorithms are still useful since they can be used for obtaining local estimates.