Simplified Analysis and Design of MIMO Ad Hoc Networks

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***Abstract*— The simple, yet powerful concept of an "erristor" and its "erristance" has been recently introduced for ad hoc networks and applied to scenarios such as retransmission (time diversity), path diversity, or a combination thereof. We extend this formalism to the case of spatial diversity, realized by employing multiple antennas at each node. Based on this framework, one can efficiently analyze and design Rayleigh-faded MIMO ad hoc networks that employ selection combining. The mathematically tractable definition of the erristor term greatly simplifies the study of a multiple-antenna network and helps solve problems based on end-to-end reliability or resource allocation easily, which we illustrate in an example. Moreover, this technique demonstrates the superiority in performance of MIMO over single-antenna routing schemes, particularly at high SNR.**

I. Introduction

In an ad hoc network, nodes are free to move randomly and organize themselves arbitrarily, and thus the network topology might change rapidly and in an unpredictable manner. Energy constraints often entail multihop routing between nodes far apart, where relays assist in the delivery of packets to their destinations [1]. While these decentralized systems are easily deployable and reconfigurable, their performance is severely susceptible to fading and interference [1], [2]. A well-known procedure to gain from fading is to employ multiple antennas at each node of the ad hoc network [3]-[6]. However, the design of multiple-input multiple-output (MIMO) ad hoc networks is a challenging task that can easily become intractable as the number of nodes in the network increases. The design problem involves allocating transmit power to each active link, to achieve at least the desired end-to-end reliability under the specified system energy constraints.

In this paper, we extend the "erristor" formalism, originally developed in [7] for single-antenna ad hoc links, to multihop MIMO systems. The erristor framework is very useful in characterizing transmissions in an ad hoc link. It can be used to greatly simplify analysis and design problems for Rayleigh-faded MIMO ad hoc networks. For systems that employ selection combining, we show that there exists a logarithmic mapping from link reliabilities to erristor values. This relationship can be exploited to effortlessly solve problems of resource (re)allocation given the network end-to-end packet delivery probability. We illustrate this with an example. The concepts developed also provide useful insights into the benefits of spatial diversity at high signal-to-noise ratios (SNRs).

II. System and Channel Model

We consider a general multihop MIMO relay network, with a single transmitting node, a single receiving node and $n-1$ relay nodes (Fig. 1). Thus, the signal from the transmitter reaches the receiver after n hops. Each node has m antennas. The channel between any two adjacent nodes is modeled as a flat (narrow-band) Rayleigh block fading channel, with an additive white Gaussian noise (AWGN) process $\mathbf{Z}$ of variance N_0. The input/output relationship at time instant t for the k^{th} link can be described as $\mathbf{Y}_t^{(k)} = \mathbf{H}_t^{(k)}\mathbf{X}_t^{(k)} + \mathbf{Z}^{(k)}$ for $k = 1 \ldots n$, where $\mathbf{X}_t^{(k)}$ and $\mathbf{Y}_t^{(k)}$ are the $(m \times 1)$ input and output vectors respectively and $\mathbf{H}_t^{(k)}$, the $(m \times m)$ channel matrix, whose entries are the large-scale path loss multiplied by the corresponding i.i.d. fading coefficients.

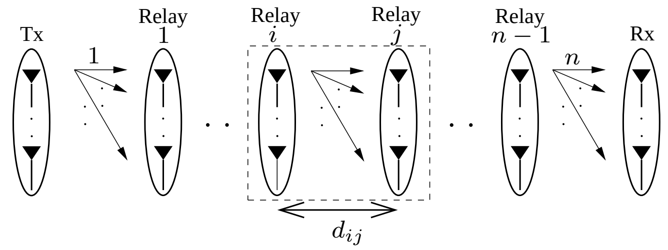

Fig. 1. The general MIMO relay model, with n links and m antennas at each node. d_{ij} is the physical separation between node i and j.

In the following discussion, we will only consider the MIMO link between node i and node j (Fig. 1). Analytical results derived for this link can be extended to other links without loss of generality.

Let the transmitter at relay i have a transmit power budget of P. Assuming that it has no knowledge of the channel state information, each of the m antennas transmits with a power level P/m in order to maximize the throughput[1] [8]. The MIMO transmission strategy aims at diversity maximization, meaning that m copies of the same signal are sent through the m antennas. Selection combining is employed at the receiver, i.e., the received signal with the maximum SNR is picked for decoding. We consider a large-scale path loss propagation

[1]With the complete knowledge of the channel (obtained usually via feedback), "waterfilling" would be employed to optimize the throughput.

model in which the transmitted power falls off with distance as $d^{-\alpha}$ [9], where α is the path loss exponent. Let the physical separation between nodes i and j be d_{ij}. This is assumed to be much greater than the antenna separation, so that the path loss is the same for all signals emanating from a node. We do not consider interference, i.e., we assume a perfect MAC scheme or light traffic for deriving our results.

III. Extension Of The Erristor Formalism

In this section, we define the erristor characterizing the transmission in the MIMO link $i \to j$. For our analysis, we employ a Rayleigh-faded model which relates transmit power, path loss and the reliability of the link[2] [7]. We also study the asymptotic behavior of the link reliability which leads us to a critical value of SNR in selection combining.

A. Erristor Modeling for the MIMO Link

We begin by noting that the power Q at each receive antenna at node j is a sum of m i.i.d. exponentials. Consequently, the pdf of Q follows the central chi-square distribution with $2m$ degrees of freedom, and has a mean $\bar{Q} = m \cdot \frac{P_0}{m} d_{ij}^{-\alpha} = P_0 d_{ij}^{-\alpha}$, where P_0 is proportional to the transmit power. We can write the cdf of the chi-square r.v. Q [10] as follows:

$$F_Q(q) = 1 - e^{-(qm/\bar{Q})} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{qm}{\bar{Q}} \right)^k, \quad q \geq 0. \qquad (1)$$

With Q_i being the signal power at the i^{th} receive antenna, the selection combining strategy picks the signal with power $S = \max(Q_1, Q_2, \ldots, Q_m)$ for decoding. The cdf of S is given by $F_S(s) = F_Q(s)^m$.

The transmission from node i to node j is successful if the maximum SNR at node j, γ_j^{max}, is greater than a certain threshold, Θ, which depends on the detector structure and the modulation and coding scheme [2]. Therefore, the reception probability is given by $p_r = \Pr[\gamma_j^{max} \geq \Theta] = \Pr[S \geq \Theta N_0]$. Thus, we get

$$p_r = 1 - \left(1 - e^{-\Theta N_0 m/\bar{Q}} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{\Theta N_0 m}{\bar{Q}} \right)^k \right)^m. \qquad (2)$$

Let R denote the normalized average NSR at the receiver, i.e., $R := \Theta N_0 / \bar{Q}$. Then, for $m \geq 1$, we have

$$\begin{aligned} p_r &= 1 - \left(1 - e^{-Rm} \sum_{k=0}^{m-1} \frac{1}{k!} (Rm)^k \right)^m \\ &= 1 - \left(e^{-Rm} \sum_{k=m}^{\infty} \frac{1}{k!} (Rm)^k \right)^m. \end{aligned} \qquad (3)$$

Fig. 2 is a plot of the reception probability (3) as a function of the number of antennas m. We remark that under good channel conditions, i.e., low values of R, the MIMO system has a higher reception probability compared to the single-antenna system, while at very low SNR values, using fewer antennas makes the link more reliable. Similar results on the sub-optimality of MIMO at low SNR have also been observed in terms of the behavior of capacity versus SNR in [13], [14] where even the receiver does not have any channel state information, and in an interference-limited environment [15].

[2]This model is preferred not only for its simplicity, but also because it overcomes some of the limitations in the "disk model" [11], [12] often used in the literature on ad hoc networks.

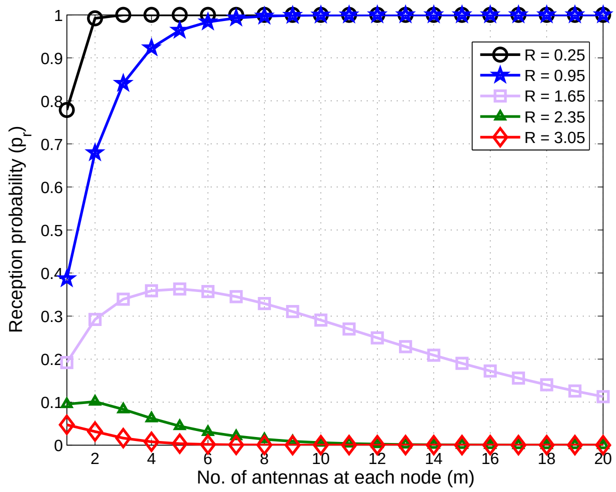

Fig. 2. Reception probabilities as a function of the number of antennas employed, for various values of the NSR R. Notice the contrast in asymptotic behavior as one set of curves approach 1, while the others tend to 0.

B. Asymptotic Behavior of the Reception Probability

We now discuss the asymptotic behavior of the reception probability as the number of antennas increases. To investigate this, first note that a Poisson distribution with parameter λ can be approximated by a Gaussian distribution with mean and variance λ, for large λ, with equality in the limiting case when $\lambda \to \infty$ [16]. Also note that the term inside the parentheses in (3) is an infinite sum over the pmf of a Poisson-distributed variable with $\lambda = Rm$. Using the Gaussian approximation for large m and nonzero R, we can approximate (3) as

$$p_r \approx 1 - \left(\frac{1}{\sqrt{(2\pi Rm)}} \int_m^{\infty} e^{-\frac{(k-Rm)^2}{2Rm}} dk \right)^m \qquad (4)$$

where we have replaced the discrete sum by a continuous integral. Writing in terms of the Q-function (area under the tail of the standard Gaussian probability density function) [9], we have

$$p_r \approx 1 - \left(Q\left(\frac{m(1-R)}{\sqrt{Rm}} \right) \right)^m. \qquad (5)$$

A well known tight upper bound for the Q-function [9] is

$$Q(x) \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}, \quad x > 0.$$

Using this, for $R < 1$, we can bound (5) as

$$p_r \gtrsim 1 - \left(\frac{\sqrt{R}}{(1-R)\sqrt{2m\pi}} \right)^m e^{-m^2(1-R)^2/2R}. \qquad (6)$$

As $m \to \infty$, the approximation becomes an equality, and thus $p_r \to 1$ as the number of antennas increases.
For $R > 1$, $(1 - R) < 0$. Using the property that $Q(-x) = 1 - Q(x)$ for $x > 0$, we can bound (5) as

$$p_r \lessapprox 1 - \left(1 - \frac{\sqrt{R}}{(R-1)\sqrt{2m\pi}} e^{-m(R-1)^2/2R}\right)^m, \quad (7)$$

which yields $p_r \to 0$ as $m \to \infty$.

Therefore, we conclude that there exists a *phase transition*, i.e., a critical value of R, $R_c = 1$ (corresponding to an average SNR of Θ), above which the reception probability is 0, and below which, success is always guaranteed, assuming infinite antennas are employed. This is, to the best of our knowledge, the first analytical derivation of this critical SNR level in selection combining.

C. Markov Approximation

To simplify the expression for p_r (3), we apply Markov's inequality [17]. The Markov bound is rather a (frequently) loose bound, but nevertheless provides valuable insights on tail probabilities. Applying it to the tail probability term gives

$$e^{-Rm} \sum_{k=m}^{\infty} \frac{1}{k!}(Rm)^k \leq \frac{Rm}{m} = R.$$

Using this in (3), we have $p_r \geq 1 - R^m$.

The deviation of the Markov bound $1 - R^m$ from the actual value (3) is plotted in Fig. 3 for different values of the parameters R and m. The threshold value of R for which the difference is less than 0.01 is calculated to be 0.103 for $m = 2$, 0.215 for $m = 3$ and 0.317 for $m = 4$. For $R > 1$, $R^m > 1$, and the approximation does not make sense.

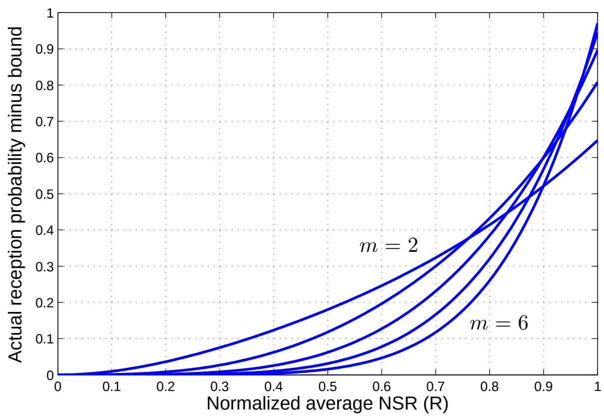

Fig. 3. The Markov approximation for p_r is tight at low values of R. However, it becomes increasingly inaccurate as R increases.

It is seen that the Markov bound is very tight only for good SNR values ($R \ll 1$) and/or when employing many antennas ($m \gg 1$). Under these conditions, first-order approximations hold, and we have

$$p_r = 1 - R^m \Leftrightarrow p_r = e^{-R^m}. \quad (8)$$

Following the erristor definition from [7, Eqn. (6)], the erristor of the MIMO link $i \to j$ is denoted by $R' = R^m$, and its value is known as its "erristance". Henceforth, we shall operate in the high SNR regime ($R \ll 1$), so that this approximated representation is accurate. Thus, we can characterize transmissions in any MIMO link by a network element, the erristor, whose value depends on the average normalized NSR at the receiving node and the number of antennas employed.

IV. Design Of MIMO Ad Hoc Networks

A powerful application of the erristor representation is the efficient design of MIMO ad hoc networks. The design problem is to set the transmit power levels at each node (or, equivalently, choose erristances) such that the end-to-end reliability, denoted by p_{EE}, is at least at the desired level p_D. In order to be able to do so, we need to find a relationship between the erristances involved with each transmission link and the equivalent erristance of the network as a whole. We discuss the two most fundamental link topologies in this regard: the multihop (series connection) and links employing time/path diversity (parallel connection).

A. Multihop Connection

Over an n-hop MIMO serial link, the end-to-end reliability is given by the product of the reception probabilities for each link. Equivalently, with R'_i denoting the erristance of the i^{th} link, we get

$$p_{EE} = e^{-\sum_{i=1}^{n} R'_i}. \quad (9)$$

The equivalent erristance R_{tot} is given by

$$R_{tot} = -\ln p_{EE} = \sum_{i=1}^{n} R'_i = \sum_{i=1}^{n} R_i^m. \quad (10)$$

B. Parallel Connection

For a more complicated network with retransmission, the transmission is successful if any one copy of the signal is successfully decoded by the receiver. Extending the result from [7, Th. 2] to a MIMO system, we conclude that erristors connected in parallel have to be multiplied, i.e.,

$$R_{tot} = \prod_{i=1}^{n} R'_i = \prod_{i=1}^{n} R_i^m. \quad (11)$$

With the knowledge of the series and parallel erristor equivalents, we can simplify most networks, since they can be put down as a combination of these two fundamental connections.

The following example describes how the erristor framework is able to reduce complex design problems to simple polynomial equations, that are analytically tractable.

Example: Consider the three-hop MIMO network in Fig. 4, where each node has m antennas and the spacing between any two adjacent nodes is equal to d. Node 1 transmits its packet twice, once to node 2 and again over the link $1 \to 3$. However, node 2 also listens when the packet is being sent to node 3. This implicit transmission is modeled by adding an erristor in parallel to link $1 \to 2$, whose value is denoted by $R_{12,i}^m$. Since $d_{13} = 2d_{12}$, it is easy to see that $R_{12,i} = 2^{-\alpha} R_{13}$. The design

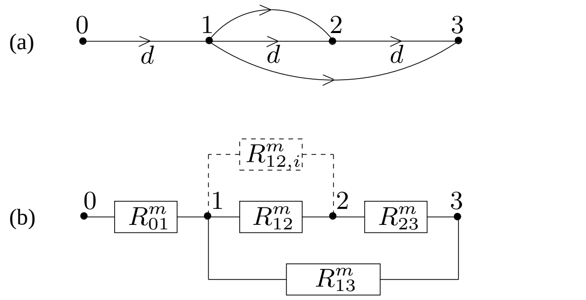

Fig. 4. An example: (a) A three-hop network, with node 1 having to transmit twice. (b) The corresponding erristor model, which is much easier to analyze. The dashed box $R^m_{12,i}$ denotes the implicit erristor.

problem is to choose the erristances to guarantee atleast the desired end-to-end reliability p_D, which is specified as 90% (say). $p_{EE} \geq p_D$ implies $R_{tot} \leq -\ln p_{EE} \approx 0.105$.

Let us assume the power required per transmission to be the same. Since the distances between adjacent nodes are the same, we require $R_{01} = R_{12} = R_{23} = R$. Also, since $d_{13} = 2d_{12}$, we need to set $R_{13} = R \cdot 2^\alpha$. Thus, $R_{tot} = R^m + (R^{2m} + R^m)(R \cdot 2^\alpha)^m$. Notice that node 2 needs to transmit twice and thereby uses up twice the power compared to the other two nodes. At $R_{tot} = 0.105$, a solution for $m = 1$ and $\alpha = 3.5$ is $R = 0.061$. For $m = 2$, $R = 0.157$ satisfies the equation. Just by increasing the number of antennas at each node by one, a (massive) 61% reduction in transmit power is observed.

Consider now a more realistic scenario where each node expends the same net transmission power. Clearly, $R_{01} = R_{23} = R$, while node 1 having to transmit with the same net transmit power requires

$$\frac{d^\alpha}{R} = \frac{d^\alpha}{R_{12}} + \frac{(2d)^\alpha}{R_{13}}.$$

One possible setting to achieve this is by letting $R_{13} = 2^\alpha R_{12}$, which results in $R_{12,i} = R_{12}$ and $R_{12} = 2R$. To meet the desired reliability, we require $R^m + ((2R)^{2m} + R^m)(2^\alpha 2R)^m \leq 0.105$. For a single-antenna system and assuming $\alpha = 3.5$, the above equation is satisfied with equality at $R = 0.048$. Likewise, for $m = 2$, $R = 0.115$, which conforms to less than $1/2$ the original power. With $m = 3$, the solution is given by $R = 0.143$, resulting in a further 20% reduction in power consumption.

This simplified procedure also lets us solve problems based on resource reallocation effortlessly. To see this, just imagine the scenario when node 2 exhausts all its energy. This would effectively make the link $1 \rightarrow 2$ useless. Our erristor network would then consist of just R^m_{01} in series with R^m_{13}. Depending on the system constraints, we can suitably design these two erristors to meet the desired requirements.

V. Comparison Of MIMO With Single-Antenna Routing Schemes

In this section, we apply the erristor formalism to compare the performance of the multihop MIMO scheme with two conventional single-antenna routing schemes (multihop routing and connections with time diversity (retransmissions)), to study the relative benefits of spatial diversity. Primarily, we focus on two important aspects: total network energy consumption (per packet) and transmission delay.

To make the comparison fair, we take the same number of total transmissions and the same end-to-end link distance d for each of the three schemes. With n transmissions and m outgoing paths from each node, the total normalized energy consumption (per packet) is easily seen to be

$$E_{tot} = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{d_{ij}^\alpha}{R_{ij}}. \tag{12}$$

Consider a MIMO network, with m antennas at each node and n hops, each of length d/n. Then, the total number of transmissions is mn. With R denoting the NSR at each receive antenna, $R_{tot} = nR^m$, and the total energy consumed

$$E_{tot} = mn\left(\frac{d}{n}\right)^\alpha \left(\frac{n}{R_{tot}}\right)^{\frac{1}{m}}. \tag{13}$$

For the single-antenna multihop scheme with mn hops, each of length $d/(mn)$, $R_{tot} = mnR'$, where R' is the NSR at each receiving node. Thus, the total energy consumption is

$$E'_{tot} = mn\left(\frac{d}{mn}\right)^\alpha \frac{mn}{R_{tot}}. \tag{14}$$

For the system with mn retransmissions, $R_{tot} = (R'')^{mn}$, where R'' denotes the NSR at the receiver, and

$$E''_{tot} = mnd^\alpha \left(\frac{1}{R_{tot}}\right)^{\frac{1}{mn}}. \tag{15}$$

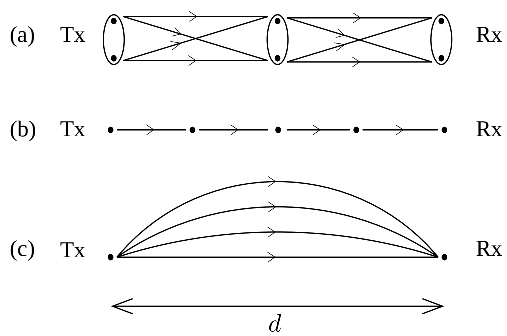

Fig. 5. The three schemes considered, with each having to transmit four times. (a) The MIMO network, employing two antennas at each node. (b) The single-antenna multihop scheme (c) Connections with retransmission involved. The end-to-end distance is d in each case.

We now compare the three routing schemes for the particular case of $n = 2$ and $m = 2$ (Fig. 5).

The ratio between the total consumed energies of MIMO and the single-antenna multihop schemes can be simplified to

$$\frac{E_{tot}}{E'_{tot}} = 2^{\alpha - \frac{3}{2}} R_{tot}^{\frac{1}{2}}. \tag{16}$$

From this, we can deduce that the spatial diversity scheme is more energy-efficient if

$$R_{tot} < 2^{3-2\alpha} \Leftrightarrow p_D > e^{-2^{(3-2\alpha)}}. \tag{17}$$

The area in the (α, p_D) plane over which MIMO networks consume less energy than multihop networks and the energy gain for various path loss exponents are plotted in Fig. 6. For practical scenarios such as high p_D and moderate α, the MIMO scheme clearly outperforms the serial link. Substantial energy gains are observed as p_D moves closer to unity.

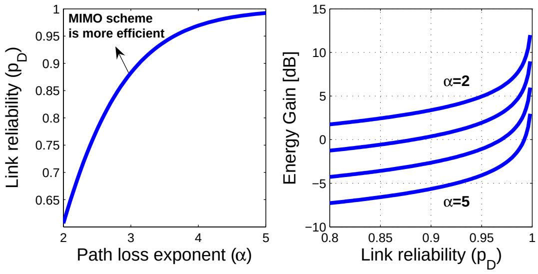

Fig. 6. (Left) MIMO is more energy-efficient than the multihop scheme in the region above the curve. (Right) The corresponding energy gain involved for different path loss exponents.

The ratio between the energies of the MIMO and the time diversity schemes is given by

$$\frac{E_{tot}}{E''_{tot}} = 2^{-\alpha+\frac{1}{2}} R_{tot}^{-\frac{1}{4}}. \tag{18}$$

From this, we can deduce that MIMO consumes less energy than the network with retransmission if

$$R_{tot} > 2^{2-4\alpha} \Leftrightarrow p_D < e^{-2^{(2-4\alpha)}}. \tag{19}$$

This curve is plotted in Fig. 7. For the retransmission scheme to outperform MIMO, $1 - p_D$ must be extremely small ($\lessapprox 10^{-3}$). This is very hard to realize, and for all practical purposes, MIMO is more energy-efficient. The plots in Figs. 6, 7 establish that huge energy gains are possible by using MIMO transmission strategies.

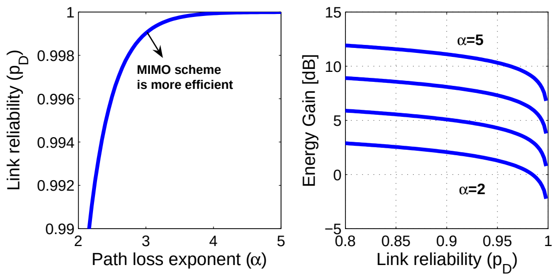

Fig. 7. (Left) The region in the (α, p_D) plane where MIMO consumes less energy than the retransmission scheme. (Right) The corresponding energy gain as a function of p_D for different path loss exponents.

Another advantage of MIMO over the single-antenna systems is in the smaller end-to-end transmission delay. Considering the schemes in Fig. 5, we see that the MIMO system can transmit in half the time required for the other two. On using m antennas, the transmission delay reduces by a factor of $1/m$. Equivalently, the MIMO system would see more independent realizations of the channel than the single-antenna systems in a given time interval, resulting in huge diversity benefits.

VI. Conclusions

The erristor formalism is developed for MIMO ad hoc networks employing selection combining, which greatly simplifies problems related to their analysis and design. With the knowledge of series and parallel erristor equivalents, complex resource (re)allocation problems reduce to simple polynomial equations, which are easier to handle, as we have demonstrated in an example. Based on the erristor framework, the MIMO multihop network is shown to be more energy-efficient and to have lower transmission delays compared to the single-antenna schemes (with or without retransmission), establishing the fact that spatial diversity can greatly benefit at high SNR values. Further, the asymptotic behavior of the MIMO link reliability as the number of antennas goes to infinity is studied, and as a useful side result, the critical value of the SNR, above which the reception probability is always one and below which it is always zero, is calculated.

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